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Stringy cosmic strings in matter coupled $N = 2$, $d = 4$ supergravity

Eric A. Bergshoeff and Jelle Hartong
Centre for Theoretical Physics, University of Groningen,
Nijenborgh 4, 9747 AG Groningen, The Netherlands
E-mail: E.A.Bergshoeff@rug.nl, J.Hartong@rug.nl

Mechthild Hübscher and Tomás Ortín
Instituto de Física Teórica UAM/CSIC,
Facultad de Ciencias C-XVI, C.U. Cantoblanco,
E-28049 Madrid, Spain
E-mail: Mechthild.Huebscher@uam.es, Tomas.Ortin@cern.ch

Abstract: We extend the system of ungauged $N = 2, d = 4$ supergravity coupled to vector multiplets and hypermultiplets with 2-form potentials. The maximal number of 2-form potentials that one may introduce is equal to the number of isometries of either the special Kähler or quaternionic Kähler sigma model. We show that the local supersymmetry algebra can be realized on the 2-form potentials. These 2-forms couple electrically to strings which we refer to as stringy cosmic strings. The 1/2 BPS bosonic world-sheet actions for these strings are constructed and we discuss the properties of the 1/2 BPS stringy cosmic string solutions.

Keywords: p-branes, Extended Supersymmetry, String theory and cosmic strings.

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Contents

1. Introduction

When constructing a matter-coupled supergravity theory one usually concentrates on the fields that describe the physical states of the theory in question. Generically the bosonic states are represented by the graviton, and a set of matter fields that generically are differential forms of low rank $(d-2)/2 \geq p \geq 0$ for d even and $(d-3)/2 \geq p \geq 0$ for d

odd, respectively. To describe the coupling to branes one is naturally led to consider the dual $(d-p-2)$-form potentials as well. For $p \neq 0$ and at leading order, the construction of the dual potentials is rather straightforward since the original low-rank differential form fields always occur via their curvatures. This means that one may even eliminate the potentials of the theory in favor of their duals. However, at higher orders, there may be non-derivative couplings and, while the dualization would still be possible, the elimination would not. A prime example of this is the trilinear coupling of the 3-form potential of $d = 11$ supergravity. In this case one can introduce a dual 6-form potential without being able to eliminate the 3-form potential. This is related to the fact that the 6-form field transforms under the gauge transformations of the 3-form potential leading to a non-trivial bosonic gauge algebra [1].

The situation is more involved for the scalar fields, i.e. $p = 0$ since often they appear via non-linear non-derivative couplings. It is instructive to consider the explicit example of IIB supergravity which has two scalars: the dilaton and the RR axion. Together they parameterize the scalar coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. The dualization of the RR axion is straightforward since at leading order it only appears under a derivative. The dual RR 8-form potential couples to the D7-brane. However, the definition of the axion is basis-dependent. Using another coordinate system for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset manifold one can define a new axion χ' which is different from the RR axion as explained in [2]. Dualizing χ', which is a function of the old dilaton and RR axion, leads to a new 8-form potential that is not related to the RR 8-form potential by any $\mathrm{SL}(2,\mathbb{R})$ duality transformation. To obtain a manifestly $\mathrm{SL}(2,\mathbb{R})$-covariant dualization prescription of all possible axions one must dualize the Noether currents associated to the presence of isometries of the scalar manifold. After all, in an appropriate coordinate system, these isometries become shift symmetries of given scalar fields. In the case of $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ there are three isometries and this procedure leads to three dual 8-form potentials. Since there are only two scalars and one cannot have more dual 8-form potentials than scalars one finds that the triplet of 8-form potentials satisfies a single duality-invariant constraint [1, 3, 4]. Another way to see this is by noting that one of the three scalars on which the isometries act as shifts does not correspond to a (discrete) isometry of the quantum moduli space $\mathrm{SL}(2,\mathbb{Z})\backslash\,\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ so that effectively only two 8-forms need to be considered.

The 8-form potentials of IIB supergravity play an important role when discussing the supersymmetry properties of 7-branes in ten dimensions [2, 5]. Likewise in four dimensions 2-form potentials are dual to those scalars which parameterize the Noether currents. They couple electrically to 1-dimensional branes which we refer to as stringy cosmic strings in analogy with the terminology used in [6] where a subset of the stringy cosmic strings of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset was studied.

In this paper we generalize the case of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset in four dimensions to $N = 2$ supergravity coupled to an arbitrary number of vector and hypermultiplets whereby we assume that the scalar sigma models admit some isometry group. This is in no way a restrictive condition because without isometries one cannot even define a 2-form potential. It was shown in [7] that one cannot in general dualize just any scalar into a 2-form potential. The objects to dualize are those Noether currents associated with the isometries of the scalar sigma models which extend to be symmetries of the full theory. Dualizing the Noether currents one obtains as many 2-forms as there are isometries. In general the field strengths of these 2-forms satisfy constraints such that the number of 2-form degrees of freedom equals the number of scalar degrees of freedom which occur in the Noether currents.

We explicitly construct the Noether currents for all the duality symmetries of ungauged $N = 2, d = 4$ supergravity coupled to both vector multiplets and hypermultiplets. Via a straightforward dualizing prescription we construct the 2-form potentials and prove that the supersymmetry algebra can be closed on them. Once we have found the explicit supersymmetry transformations for the 2-forms we proceed to construct the leading terms of a half-supersymmetric world-sheet effective action. Finally we discuss to some detail the properties of the half-supersymmetric stringy cosmic string solutions. The above program is first performed for the duality symmetries associated with the scalars coming from the vector multiplets and then repeated for the duality symmetries associated with the scalars coming from the hypermultiplets.

In dualizing the 2-forms which are dual to the scalars of the vector multiplets it turns out to be necessary to incorporate into the discussion both the 1-forms and their duals. This is because the gauge transformations of the 2-forms involve both the 1-forms and their duals. We will therefore also briefly discuss the supersymmetry properties of the dual 1-forms and as a side result construct world-line effective actions for 0-branes carrying an arbitrary number of electric and magnetic charges. These 0-brane effective actions may be used as sources for extreme supersymmetric black holes with electric and magnetic charges.

This paper is organized as follows. In section 2 we give a brief description of $N = 2, d = 4$ supergravity coupled to vector multiplets and hypermultiplets. In section 3 we study dual 1-forms and their supersymmetry transformation rules. These are used in section 4 to construct symplectic-invariant 0-brane word-line actions. The symplectic invariance refers to the fact that the world-line actions contain both the 1-forms and their duals. In section 5 we construct the 2-forms dual to the scalars of the vector multiplets in three steps. In section 5.1 we construct the Noether current 1-forms associated to the isometries of the special Kähler manifold. They are on-shell dualized into 2-forms in section 5.2. The supersymmetry transformations of these 2-forms are constructed in section 5.3. In section 6 we will apply our results to construct the stringy cosmic string world-sheet effective actions. The supersymmetric stringy cosmic string solutions associated to these effective actions are discussed in section 7. In sections 8 to 10 we repeat this program for the isometries of the quaternionic Kähler manifold which lead to the 2-forms dual to the hyperscalars. Our conclusions are contained in section 11.

2. Matter-coupled, ungauged, $N = 2$, $d = 4$ supergravity

Our starting point is $N = 2, d = 4$ ungauged supergravity coupled to n_V vector and n_H hypermultiplets. This is the same theory that was studied in [8], whose conventions we use here.[1] In this section we will briefly review it for the sake of self-consistency, referring the

[1] They are those of ref. [10] with some minor changes introduced in refs. [8, 9].

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reader to [8, 9], the reviews [10, 11] and the original papers [12, 13] for more details. Our conventions have been summarized in appendix A.

The bosonic fields of the theory are those of the $N = 2$, $d = 4$ supergravity multiplet (metric and graviphoton) and of n_V vector multiplets (n_V complex scalars and n_V vectors) and n_H hypermultiplets ($4n_H$ real scalars). The graviphoton together with the n_V vectors are combined into the vector A^{Λ}_{μ} where $\Lambda = 0, 1, \ldots, n_V$. The complex scalars will be denoted by Z^i with $i = 1, \ldots, n_V$ while the real scalars will be denoted by q^u with $u = 1, \ldots, 4n_H$.

The action of the bosonic fields of the theory is

$$\begin{aligned} S = \int d^4x \sqrt{|g|}\, \big[R &+ 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\mathsf{H}_{uv} \partial_\mu q^u \partial^\mu q^v \\ &+ 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu\star} F^{\Sigma}{}_{\mu\nu} \big] \,, \end{aligned} \tag{2.1}$$

where the complex scalars Z^i parameterize a special Kähler manifold and where the real scalars q^u parameterize a quaternionic Kähler manifold. For their definitions and properties we refer the reader to appendices B and C. The metric on the special Kähler manifold is denoted by $\mathcal{G}_{ij^*}$, where the index $(j^*)i$ is a (anti-)holomorphic index. The field strengths of the vectors A^{Λ}_{μ} are $F^{\Lambda}_{\mu\nu} = \partial_\mu A^{\Lambda}_{\nu} - \partial_\nu A^{\Lambda}_{\mu}$. The scalars couple to the vectors via the period matrix $\mathcal{N}_{\Lambda\Sigma}$ whose definition is given in appendix B. The last term in (2.1) is topological with

$$^{\star}F^{\Sigma}{}_{\mu\nu} \equiv \tfrac{1}{2\sqrt{|g|}}\, \epsilon_{\mu\nu\rho\sigma} F^{\Sigma\,\rho\sigma} \,. \tag{2.2}$$

It is an important feature of the above action that the period matrix $\mathcal{N}$ is only a function of the complex scalars Z^i and Z^{*i^*} of the vector multiplets and does not depend on the quaternionic scalars q^u of the hypermultiplets. The vector and hypermultiplets only interact gravitationally.

The field strengths $F^{\Lambda}{}_{\mu\nu}$ of the vector potentials $A^{\Lambda}{}_{\mu}$ satisfy the Bianchi identity

$$\nabla_\nu ({}^{\star}F^{\Lambda})^{\nu\mu} = 0 \qquad \text{or} \qquad dF^{\Lambda} = 0 \,, \tag{2.3}$$

and the equation of motion

$$\frac{1}{8\sqrt{|g|}} \frac{\delta S}{\delta A^{\Lambda}_{\mu}} = \nabla_\nu ({}^{\star}F_{\Lambda})^{\nu\mu} = 0 \,, \tag{2.4}$$

where we have defined the dual vector field strength F_{Λ} by

$$F_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}} \frac{\delta S}{\delta {}^{\star}F^{\Lambda}{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Sigma}{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^{\star}F^{\Sigma}{}_{\mu\nu} \,. \tag{2.5}$$

The equation of motion (2.4) can be interpreted as a Bianchi identity for the dual field strength F_{Λ},

$$dF_{\Lambda} = 0 \,, \tag{2.6}$$

implying the local existence of $n_V + 1$ dual vector fields A_{Λ}, i.e. locally $F_{\Lambda} = dA_{\Lambda}$. The equation of motion and Bianchi identity for A^{Λ}, eqs. (2.4) and (2.3), respectively, can be summarized as

$$d\mathcal{F} = 0 \,, \tag{2.7}$$

where $\mathcal{F}$ is the $(2n_V+2)$-dimensional vector of field strengths

$$\mathcal{F} \equiv \begin{pmatrix} F^\Lambda \\ F_\Lambda \end{pmatrix} . \tag{2.8}$$

The Maxwell equations and Bianchi identities are left (formally) invariant by the transformations of the vector field strengths

$$\mathcal{F}' = \mathcal{S}\mathcal{F} , \qquad \mathcal{S} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_V+2,\mathbb{R}) , \tag{2.9}$$

A, B, C and D being $(n_V+1)\times(n_V+1)$ matrices. The $(2n_V+2)$-dimensional vector of potentials

$$\mathcal{A} \equiv \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix} , \tag{2.10}$$

whose local existence is implied by eqs. (2.7), transforms in the same way. However, since the dual potentials, A_Λ, depend in a non-local way on the 'fundamental' ones, A^Λ, these transformations are non-local and are not symmetries of the action, which only depends on the fundamental potentials, but only of the Maxwell equations and Bianchi identities.

We have to take into account, however, that the definition of the dual field strength F_Λ involves the period matrix $\mathcal{N}_{\Lambda\Sigma}$. In order to preserve this relation, the period matrix must transform under the above $GL(2n_V+2,\mathbb{R})$ transformations as

$$\mathcal{N}' = (D\mathcal{N}+C)(B\mathcal{N}+A)^{-1} . \tag{2.11}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is symmetric in its indices Λ and Σ. Demanding that this symmetry is preserved under the transformation (2.11) one finds that the matrices A, B, C, D must satisfy

$$D^T B = B^T D , \quad C^T A = A^T C \quad \text{and} \quad D^T A - B^T C = \mathbb{1} , \tag{2.12}$$

or

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{with} \quad \Omega \equiv \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{2.13}$$

so that $\mathcal{S} \in \mathrm{Sp}(2n_V+2,\mathbb{R})$ and only this subgroup of elements $\mathcal{S} \in GL(2n_V+2,\mathbb{R})$ can be a symmetry of all the equations of motion of the theory.[2]

It can be checked that this condition is enough for the transformations to leave invariant the Einstein equations as well, but, to be symmetries of all the equations of motion, they have to leave invariant the scalar equations of motion as well.

Since the period matrix is a function of the complex scalars, $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$, the transformations (2.11) induce transformations of the complex scalars Z^i. The kinetic term for Z^i in (2.1) will be invariant when the scalar transformations (2.11) are isometries of the metric $\mathcal{G}_{ij^*}$. Thus, out of the group $GL(2n_V+2,\mathbb{R})$, only the subgroup G_V of isometries of

[2]This, in fact, is the largest possible electro-magnetic duality group of any Lagrangian depending on Abelian field strengths, scalars and derivatives of scalars as well as spinor fields [14].

the special Kähler manifold that can be embedded in $\mathrm{Sp}(2n_V+2,\mathbb{R})$ is a symmetry of the full set of equations of motion and Bianchi identities. In order for G_V to be a symmetry of the complete supergravity theory, it must satisfy some extra conditions that we will study in section 5.1, see (5.31). There can be further symmetries which are the isometries of the quaternionic Kähler manifold, i.e. isometries of the metric H_{uv}. These isometries are unrelated to the electromagnetic duality group $\mathrm{Sp}(2n_V+2,\mathbb{R})$. All these symmetries and the extended objects associated to them will be the subject of this paper.

The fermionic fields of the theory are those of the $N=2$, $d=4$ supergravity multiplet (two gravitini $\Psi_{I\,\mu}\,, I=1,2$), n_V vector multiplets (n_V gaugini $\lambda^{i\,I}$) and of n_H hypermultiplets ($2n_H$ hyperini $\zeta_\alpha\,, \alpha=1,\dots,2n_H$). We take all spinors to be complex Weyl spinors. We define $\lambda^{i^*}{}_I=(\lambda^{i\,I})^*$ and $\zeta^\alpha=(\zeta_\alpha)^*$. The index α is an $\mathrm{Sp}(2n_H)$ index where by $\mathrm{Sp}(2n_H)$ we mean the compact symplectic group $\mathrm{Sp}(2n_H)\simeq \mathrm{U}(4n_H)\cap \mathrm{Sp}(4n_H,\mathbb{C})$.

The R-symmetry group of $N=2$, $d=4$ supergravity is $\mathrm{SU}(2)\times \mathrm{U}(1)$. The $\mathrm{U}(1)$ gauge connection is the Kähler connection 1-form, denoted by $\mathcal{Q}$, and the spinors all carry a particular Kähler weight with respect to $\mathcal{Q}$ (see appendix B for more details). The $\mathrm{SU}(2)$ gauge connection is denoted by $\mathsf{A}_I{}^J$ and acts on all objects which carry an $\mathrm{SU}(2)$ index $I=1,2$ (see appendix C for more details about $\mathsf{A}_I{}^J$).

From this point on we will refer to the upper case Greek indices as symplectic indices and to vectors X given by

$$X=\begin{pmatrix} X^\Lambda \\ X_\Lambda \end{pmatrix} \tag{2.14}$$

as symplectic vectors. Given two symplectic vectors X and Y we define the symplectic-invariant inner product, $\langle X \mid Y\rangle$, by

$$\langle\, X \mid Y\,\rangle = X^T\Omega Y = X_\Lambda Y^\Lambda - X^\Lambda Y_\Lambda\,. \tag{2.15}$$

When writing forms inside a symplectic inner product we will implicitly assume that we are taking the exterior product of both. One should then keep in mind that $\langle X_{(p)} \mid Y_{(q)}\rangle = (-1)^{pq+1}\langle Y_{(q)} \mid X_{(p)}\rangle$, where $X_{(p)}$ and $Y_{(q)}$ are p- and q-forms, respectively. Later in section 5 we will encounter symplectic inner products of the form $\langle X_{(p)} \mid T\,Y_{(q)}\rangle$ where T is a generator of $\mathfrak{sp}(2n_V+2,\mathbb{R})$ satisfying $T^T\Omega+\Omega T=0$ with Ω as in eq. (2.13). For such inner products we have the property $\langle X_{(p)} \mid T\,Y_{(q)}\rangle = (-1)^{pq}\langle Y_{(q)} \mid T\,X_{(p)}\rangle$.

We next discuss the supersymmetry transformations of all the fields of the theory. To lowest order in fermions, the supersymmetry transformations of the bosonic fields are

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\frac{i}{4}\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \text{c.c.}\,, && (2.16)\\
\delta_\epsilon A^\Lambda{}_\mu &= \frac{1}{4}\mathcal{L}^\Lambda\,\epsilon_{IJ}\bar{\psi}^I_\mu\epsilon^J + \frac{i}{8}\mathfrak{D}_i\mathcal{L}^\Lambda\,\epsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + \text{c.c.}\,, && (2.17)\\
\delta_\epsilon Z^i &= \frac{1}{4}\bar{\lambda}^{Ii}\epsilon_I\,, && (2.18)\\
\delta_\epsilon q^u &= \frac{1}{4}\mathsf{U}^{\alpha I\,u}\bar{\zeta}_\alpha\epsilon_I + \text{c.c.}\,, && (2.19)
\end{aligned}$$

where $\mathcal{L}^\Lambda$ is defined in appendix B as the upper part of the symplectic section $\mathcal{V}$ in terms of which a special Kähler manifold can be defined and where $\mathfrak{D}_i\mathcal{L}^\Lambda$ is the Kähler-covariant derivative of $\mathcal{L}^\Lambda$ on the special Kähler manifold. The object $\mathsf{U}^{\alpha I\, u}$ which appears in eq. (2.19) is the complex conjugate of the so-called inverse Quadbein, i.e. $\mathsf{U}^{\alpha I\, u} = (\mathsf{U}_{\alpha I}{}^{u})^*$. A Quadbein, denoted by $\mathsf{U}^{\alpha I}{}_u$, is a Vielbein of the quaternionic Kähler manifold and is defined in appendix C. The index pair αI on a Quadbein originates from the fact that the holonomy group of a quaternionic Kähler manifold is $\mathrm{Sp}(1)\times\mathrm{Sp}(2n_H)$ with $\mathrm{Sp}(1)\simeq\mathrm{SU}(2)$. The index pair αI is raised and lowered under complex conjugation, e.g. $\mathsf{U}_{\alpha I\, u} = (\mathsf{U}^{\alpha I}{}_u)^*$.

The fermionic field supersymmetry transformations are

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ} T^+{}_{\mu\nu}\gamma^\nu \epsilon^J\,, \tag{2.20}$$

$$\delta_\epsilon \lambda^{iI} = i\, \not{\partial} Z^i \epsilon^I + \epsilon^{IJ}\, \not{G}^{i\,+}\epsilon_J\,. \tag{2.21}$$

$$\delta_\epsilon \zeta_\alpha = i \mathsf{U}_{\alpha I\, u}\, \not{\partial} q^u \epsilon^I\,, \tag{2.22}$$

The derivative $\mathfrak{D}_\mu$ is the Lorentz, Kähler and SU(2) covariant derivative acting on objects with nonzero Kähler weights and SU(2) indices I, J. In particular, it acts on the local supersymmetry transformation parameter ϵ_I as

$$\mathfrak{D}_\mu \epsilon_I = (\nabla_\mu \;+\; \frac{i}{2}\, \mathcal{Q}_\mu)\; \epsilon_I \;+\; \mathsf{A}_{\mu\, I}{}^{J}\; \epsilon_J\,, \tag{2.23}$$

where $\mathcal{Q}_\mu$ is the pullback of the Kähler connection defined in eq. (B.2) and where $\mathsf{A}_{\mu\, I}{}^{J}$ is the pull back of the SU(2) connection $\mathsf{A}_I{}^J$ of the quaternionic-Kähler manifold,

$$\mathsf{A}_{\mu\, I}{}^{J} = \mathsf{A}_{u\, I}{}^{J}\partial_\mu q^u\,. \tag{2.24}$$

In the variation of the gravitini the hyperscalars only appear via the SU(2) connection $\mathsf{A}_{\mu\, I}{}^{J}$, while in the variation of the gaugini the hyperscalars do not appear at all. The 2-forms T^+ and $G^{i\,+}$ appearing in eqs. (2.20) and (2.21) are the self-dual parts of the graviphoton and matter vector field strengths, respectively. They can be written in a manifestly symplectic-invariant form as

$$T^+ \;=\; \langle\, \mathcal{V} \mid \mathcal{F}\,\rangle\,, \tag{2.25}$$

$$G^{i\,+} \;=\; \frac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathfrak{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}\,\rangle\,. \tag{2.26}$$

The commutator of two supersymmetry transformations on the bosonic p-form fields presented in this section, i.e. scalars and 1-forms, has the universal form[3]

$$[\delta_\eta, \delta_\epsilon] = \delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda)\,, \tag{2.27}$$

where $\delta_{\text{g.c.t.}}(\xi)$ is an infinitesimal general coordinate transformation with parameter ξ^μ and $\delta_{\text{gauge}}(\Lambda)$ is a U(1) gauge transformation with parameter Λ^Λ. The parameters ξ^ρ and Λ^Λ are given by the spinor bilinears

$$\xi^\mu \;\equiv\; -\frac{i}{4}\bar{\eta}^I\gamma^\mu\epsilon_I + \text{c.c.}\,, \tag{2.28}$$

$$\Lambda^\Lambda \;\equiv\; -\xi^\rho A^\Lambda_\rho + \frac{1}{4}\left(\mathcal{L}^\Lambda \epsilon_{IJ}\bar{\eta}^I\epsilon^J + \text{c.c.}\right)\,. \tag{2.29}$$

[3] On the Vierbein and the spinors the commutator also contains a local Lorentz transformation that does not act on the p-forms since these carry no Lorentz vector or spinorial indices.

In the next sections we will define new dual fields of $N = 2, d = 4$ supergravity which will satisfy the same universal algebra with the possible addition of specific gauge transformations which do not act on the original 'fundamental' fields that we have introduced in this section.

3. The 1-forms

The $N = 2, d = 4$ supergravity theory coupled to n_V vector multiplets contains $n_V + 1$ 'fundamental' vector fields $A^{\Lambda}{}_{\mu}$ whose supersymmetry transformation rules are given in eq. (2.17). The potentials $A^{\Lambda}{}_{\mu}$ couple electrically to charged particles. In the next section we will construct the leading terms of the bosonic part of the κ-symmetric world-line effective actions for particles electrically charged under $A^{\Lambda}{}_{\mu}$.

As we mentioned in section 2, the equations of motion of the potentials $A^{\Lambda}{}_{\mu}$, eqs. (2.4), can be understood as providing the Bianchi identities for a set of dual field strengths F_{Λ} defined in eq. (2.5). These equations imply the on-shell local existence of $n_V + 1$ dual potentials $A_{\Lambda\,\mu}$. The dual potentials $A_{\Lambda\,\mu}$ couple electrically to particles which are magnetically charged under the fundamental vector fields $A^{\Lambda}{}_{\mu}$. In this section we will derive the supersymmetry transformation rules for the dual potentials $A_{\Lambda\,\mu}$. This result will then be used in the next section to construct the leading terms of the bosonic part of the κ-symmetric world-line effective actions for particles electrically charged under the $A_{\Lambda\,\mu}$.

The fundamental potentials and their duals can be seen as, respectively, the upper and lower components of the symplectic vector $\mathcal{A}_{\mu}$ defined in eq. (2.10). Electric-magnetic duality transformations act linearly on it. This suggests the following Ansatz for the supersymmetry transformation rule of $\mathcal{A}$:

$$\delta_{\epsilon}\mathcal{A}_{\mu} \;=\; \frac{1}{4}\mathcal{V}\,\epsilon_{IJ}\bar{\psi}^{I}_{\mu}\epsilon^{J} + \frac{i}{8}\mathfrak{D}_{i}\mathcal{V}\,\epsilon_{IJ}\bar{\lambda}^{Ii}\gamma_{\mu}\epsilon^{J} + \text{c.c.}\,. \tag{3.1}$$

This Ansatz agrees with the supersymmetry transformation rule of the fundamental potentials $A^{\Lambda}{}_{\mu}$ as given in eq. (2.17) and with the fact that the $A^{\Lambda}{}_{\mu}$ transform linearly under $\mathrm{Sp}(2n_V + 2, \mathbb{R})$. Indeed, the supersymmetry algebra closes on the symplectic vector of 1-forms $\mathcal{A}_{\mu}$ with the above supersymmetry transformation rule. We find for the commutator of two supersymmetries acting on $\mathcal{A}_{\mu}$,

$$[\delta_{\eta}, \delta_{\epsilon}]\mathcal{A}_{\mu} = \delta_{\text{g.c.t.}}(\xi)\mathcal{A}_{\mu} + \delta_{\text{gauge}}(\Lambda)\mathcal{A}_{\mu}\,. \tag{3.2}$$

The general coordinate transformation of $\mathcal{A}_{\mu}$ is given by

$$\delta_{\text{g.c.t.}}(\xi)\mathcal{A}_{\mu} = \pounds_{\xi}\mathcal{A}_{\mu} = \xi^{\nu}\partial_{\nu}\mathcal{A}_{\mu} + (\partial_{\mu}\xi^{\nu})\mathcal{A}_{\nu}\,, \tag{3.3}$$

with $\pounds_{\xi}$ denoting the Lie derivative and where the infinitesimal parameter ξ^{ρ} is given in eq. (2.28). The gauge transformation of $\mathcal{A}_{\mu}$ is given by

$$\delta_{\text{gauge}}(\Lambda)\mathcal{A}_{\mu} = \partial_{\mu}\Lambda\,, \tag{3.4}$$

where the gauge transformation parameter Λ is the symplectic-covariant generalization of Λ^Λ as given in eq. (3.5) and is given by

$$\Lambda \equiv -\xi^\rho \mathcal{A}_\rho + \frac{1}{4}\left(\mathcal{V}\epsilon_{IJ}\bar{\eta}^I\epsilon^J + \text{c.c.}\right) . \tag{3.5}$$

4. World-line actions for 0-branes

In this section we will construct the leading terms of the bosonic part of a κ-invariant world-line effective action for 0-branes that couple to the 1-form potentials $A^\Lambda{}_\mu$ and $A_{\Lambda\,\mu}$. In doing so we will take into account the symplectic structure of the theory. The actions will be invariant under symplectic transformations provided we also transform an appropriate set of the charges, in the spirit of ref. [15].

It is clear that the 0-branes of $N = 2, d = 4$ supergravity coupled to n_V vector multiplets can carry both electric charges q_Λ and magnetic charges p^Λ with respect to the fundamental potentials $A^\Lambda{}_\mu$. The couplings of the magnetic 0-branes are, however, better described as electric couplings to the dual potentials $A_{\Lambda\,\mu}$. A 0-brane with symplectic charge vector

$$q \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} . \tag{4.1}$$

will couple electrically to the potential $\mathcal{A}$. The only symplectic-invariant coupling is $\langle q \mid \mathcal{A}\rangle$. We thus propose the following Wess-Zumino term

$$\int d\tau\, \langle\, q \mid \mathcal{A}_\mu \,\rangle \frac{dX^\mu}{d\tau} , \tag{4.2}$$

where τ is the world-line parameter and X^μ the embedding coordinate of the 0-brane. This Ansatz is clearly the only one satisfying the requirements of symplectic invariance and gauge invariance.

The corresponding kinetic term in the 0-brane action is not much more difficult to guess. Symplectic invariance requires that the charges q_Λ and p^Λ appear in a symplectic invariant combination with the scalars in the tension. The simplest combination is just the central charge

$$\mathcal{Z} = \langle\, q \mid \mathcal{V}\,\rangle , \tag{4.3}$$

whose asymptotic absolute value is known to give the mass of supersymmetric black holes of these theories. Then, the world-line effective action takes the form

$$S = \int d\tau\, |\mathcal{Z}| \sqrt{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} g_{\mu\nu}(X)} + \int d\tau \langle\, q \mid \mathcal{A}_\mu\,\rangle \frac{dX^\mu}{d\tau} . \tag{4.4}$$

Using the supersymmetry transformations (2.16), (2.18) and (3.1) we find that the action (4.4) preserves half of the supersymmetries with the projector given by

$$\epsilon_I + i\frac{\mathcal{Z}}{|\mathcal{Z}|}\epsilon_{IJ}\frac{\gamma_\tau}{\sqrt{g_{\tau\tau}}}\epsilon^J = 0 , \tag{4.5}$$

where the subindex τ means contraction of a space-time index μ with $dX^\mu/d\tau$. This is the same constraint that the Killing spinors of supersymmetric $N=2, d=4$ black holes satisfy [8, 16−18]. In the static gauge, $\dot{X}^\mu = dX^\mu/d\tau = \delta^\mu{}_t$, assuming a static metric, so that $\sqrt{g_{tt}} = e^0{}_t$ and denoting by $e^{i\alpha}$ the phase of the central charge $\mathcal{Z}$, the above projector takes the form

$$\epsilon_I + ie^{i\alpha}\epsilon_{IJ}\gamma_0\epsilon^J = 0\,. \tag{4.6}$$

This equation is satisfied for spinors of the form

$$\epsilon_I = |X|^{1/2}e^{\frac{i}{2}\alpha}\epsilon_{I\,0}\,, \qquad \epsilon_{I\,0} + i\epsilon_{IJ}\gamma_0\epsilon^{J\,0} = 0\,, \tag{4.7}$$

in which the $\epsilon_{I\,0}$ are constant spinors and with $|X|$ some real function.

5. The 2-forms: the vector case

In this section we will construct the most general 2-forms associated to the isometries of the special Kähler manifold one can introduce in $N=2, d=4$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. The 2-forms associated to the isometries of the quaternionic Kähler manifold will be discussed in section 8. For the subset of commuting isometries a similar program has been performed in [19] where also actions for the dualized scalars, which are part of so-called vector-tensor multiplets, are given.

5.1 The Noether current

As explained in section 2 only the group G_V of isometries of the special Kähler manifold which can be embedded in $\mathrm{Sp}(2n_V+2,\mathbb{R})$ are symmetries of the full set of equations of motion and Bianchi identities. Despite the fact that these duality transformations only leave invariant the equations of motion together with the Bianchi identities, it is possible to construct a conserved Noether current associated to this invariance [14]. This is because under variations of the scalars $\delta_Z\mathcal{L}+\delta_{Z^*}\mathcal{L}$ the Lagrangian is invariant up to the divergence of an anomalous current, denoted here and in [14] by $\hat{J}^\mu$. Hence, we have

$$\delta_Z\mathcal{L} + \delta_{Z^*}\mathcal{L} = -\partial_\mu(\sqrt{|g|}\hat{J}^\mu)\,. \tag{5.1}$$

In the case of p-brane actions coupled to supergravity the Noether current associated to the super-Poincaré invariance of the coupled system contains a similar anomalous contribution [20], which is known to give rise to central charges in the supersymmetry algebra.

Applying the Noether theorem we get

$$\partial_\mu\left(\delta Z^i\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})}\right) = -\partial_\mu(\sqrt{|g|}\hat{J}^\mu)\,, \tag{5.2}$$

so that the Noether current

$$J_N^\mu = \delta Z^i\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})} + \hat{J}^\mu\,, \tag{5.3}$$

is covariantly conserved, i.e. $\nabla_\mu J_N^\mu = 0$. In this Subsection we will compute J_N^μ for the isometries of the Kähler metric $\mathcal{G}_{ij^*}$ which are embedded in $\mathrm{Sp}(2n_V+2,\mathbb{R})$.

Infinitesimally, the symmetries under consideration act on the complex scalars as

$$\delta Z^i = \alpha^A k_A{}^i(Z)\,, \tag{5.4}$$

where the $k_A{}^i(Z)$ are $\dim G_V$ holomorphic Killing vectors[4] ($A = 1, \cdots, \dim G_V$) and where α^A denotes a set of real infinitesimal parameters. The Lie brackets of the Killing vectors give the Lie algebra of G_V with structure constants $f_{AB}{}^C$,

$$[k_A, k_B] = -f_{AB}{}^C k_C\,, \tag{5.5}$$

where $k_A = k_A{}^i\partial_i + k_A{}^{*\,i^*}\partial_{i^*}$.

On the vector field strengths the symmetries act as an infinitesimal $\mathrm{Sp}(2n_V + 2, \mathbb{R})$ transformation

$$\delta\mathcal{F} = T\mathcal{F}\,, \tag{5.6}$$

where $T \in \mathfrak{sp}(2n_V + 2, \mathbb{R})$, i.e. $T^T\Omega + \Omega T = 0$. The matrix T can be expressed as a linear combination of the generators of the isometry group G_V of $\mathcal{G}_{ij^*}$ that is embedded in $\mathfrak{sp}(2n_V + 2, \mathbb{R})$. In other words,

$$T = \alpha^A T_A\,, \qquad [T_A, T_B] = f_{AB}{}^C T_C\,, \qquad T_A \in \mathfrak{sp}(2n_V + 2, \mathbb{R})\,. \tag{5.7}$$

On the other hand, if

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{5.8}$$

then, the condition $T^T\Omega + \Omega T = 0$ implies

$$c^T = c\,, \quad b^T = b\,, \quad \text{and} \quad a^T = -d\,. \tag{5.9}$$

To find the current $\hat{J}^\mu$ we start by writing the Lagrangian of (2.1) in the following form

$$\mathcal{L} = \frac{1}{2}F^\Lambda{}_{\mu\nu}\frac{\partial\mathcal{L}}{\partial F^\Lambda{}_{\mu\nu}} + \mathcal{L}_{\text{inv}}\,, \tag{5.10}$$

where

$$\mathcal{L}_{\text{inv}} = \sqrt{|g|}\left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*j^*}\right]\,, \tag{5.11}$$

is the part of the Lagrangian that is invariant under (5.4) and where

$$\frac{\partial\mathcal{L}}{\partial F^\Lambda{}_{\mu\nu}} = -4\sqrt{|g|}\star F_\Lambda{}^{\mu\nu}\,. \tag{5.12}$$

Next we compute the variation of $\mathcal{L}$ with respect to the variation of the scalars

$$\delta_Z\mathcal{L} + \delta_{Z^*}\mathcal{L} = \delta\mathcal{L} - \delta_F\mathcal{L}\,, \tag{5.13}$$

where $\delta\mathcal{L}$ is the total variation and $\delta_F\mathcal{L}$ denotes the variation of $\mathcal{L}$ with respect to the field strength $F^\Lambda_{\mu\nu}$. The total variation of $\mathcal{L}$ under the transformations (5.4) and (5.6) is

$$\delta\mathcal{L} = \delta\left(-2\sqrt{|g|}F^\Lambda{}_{\mu\nu}\star F_\Lambda{}^{\mu\nu}\right) = -2\sqrt{|g|}\left[\star F_\Lambda{}^{\mu\nu}b^{\Lambda\Sigma}F_{\Sigma\,\mu\nu} + \star F^{\Lambda\,\mu\nu}c_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu}\right]\,, \tag{5.14}$$

[4]The holomorphicity of the components $k_A{}^i$ follows from the Killing equation.

where we have used eqs. (5.9). The variation, $\delta_F \mathcal{L}$, is

$$\delta_F \mathcal{L} = \delta F^{\Lambda}{}_{\mu\nu} \frac{\partial \mathcal{L}}{\partial F^{\Lambda}{}_{\mu\nu}} = -4\sqrt{|g|}\left[\star F_{\Lambda}{}^{\mu\nu} a^{\Lambda}{}_{\Sigma} F^{\Sigma}{}_{\mu\nu} + \star F_{\Lambda}{}^{\mu\nu} b^{\Lambda\Sigma} F_{\Sigma\,\mu\nu}\right] . \tag{5.15}$$

Using once again eqs. (5.9) it then follows that

$$\delta \mathcal{L} - \delta_F \mathcal{L} = 2\sqrt{|g|}\langle \star\mathcal{F}^{\mu\nu} \mid T\mathcal{F}_{\mu\nu} \rangle . \tag{5.16}$$

The result eq. (5.16) can be written (on-shell) as the divergence of an anomalous current $\hat{J}$ i.e. one can show, using eqs. (2.3) and (2.4), that

$$-\partial_\mu(\sqrt{|g|}\hat{J}^\mu) = \delta\mathcal{L} - \delta_F\mathcal{L} , \tag{5.17}$$

where $\hat{J}^\mu$ is given by

$$\hat{J}^\mu = -4\langle \star\mathcal{F}^{\mu\nu} \mid T\mathcal{A}_\nu \rangle . \tag{5.18}$$

At the same time we have for the right hand-side of this equation

$$\delta\mathcal{L} - \delta_F\mathcal{L} = \delta_Z\mathcal{L} + \delta_{Z^*}\mathcal{L} = \partial_\mu\left(\delta Z^i \frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})}\right) , \tag{5.19}$$

so that the Noether current, J^μ_N, is given by

$$J^\mu_N = \delta Z^i \frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} + \delta Z^{*i^*}\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^{*i^*})} + \hat{J}^\mu , \tag{5.20}$$

with $\hat{J}^\mu$ given by eq. (5.18), and satisfies

$$\partial_\mu\left(\sqrt{|g|}J^\mu_N\right) = 0 . \tag{5.21}$$

Under gauge transformations of the 1-form potentials $\mathcal{A}$ the anomalous current $\hat{J}^\mu$ and hence J^μ_N are not invariant: they transform as the divergence of an anti-symmetric tensor. We will have to take this point into account in the next subsection when dualizing the Noether current into a 2-form.

It will be convenient to write the scalar part of the Noether current, i.e. the part $J_N - \hat{J}$, in terms of the symplectic sections $\mathcal{V}$ instead of the physical scalars since $\mathcal{V}$ transforms linearly under $\mathrm{Sp}(2n_V + 2, \mathbb{R})$. This is achieved using

$$\delta\mathcal{V} = \delta Z^i \partial_i \mathcal{V} + \delta Z^{*i^*}\partial_{i^*}\mathcal{V} , \tag{5.22}$$

and eqs. (B.8) and (B.9). We have

$$\delta Z^i \frac{\partial\mathcal{L}}{\partial(\partial_\mu Z^i)} = -2i\sqrt{|g|}\langle \delta\mathcal{V} \mid \mathfrak{D}^\mu\mathcal{V}^* \rangle . \tag{5.23}$$

Hence, the Noether current (5.20) can be expressed in terms of $\mathcal{V}$ as

$$J^\mu_N = -2i\langle \delta\mathcal{V} \mid \mathfrak{D}^\mu\mathcal{V}^* \rangle + \text{c.c.} + \hat{J}^\mu . \tag{5.24}$$

We continue to find an explicit expression for $\delta\mathcal{V}$. The symplectic sections transform under global $\mathrm{Sp}(2n_V+2,\mathbb{R})$ and under local Kähler transformations. The Kähler potential transforms as

$$\delta_\alpha\mathcal{K} \equiv \pounds_{\alpha^A k_A}\mathcal{K} = \alpha^A\left(k_A{}^i\partial_i\mathcal{K} + k_A{}^{*\,i^*}\partial_{i^*}\mathcal{K}\right) = \lambda(Z)+\lambda^*(Z^*)\,, \qquad \lambda(Z)=\alpha^A\lambda_A(Z)\,. \tag{5.25}$$

It can be shown that the functions $\lambda_A(Z)$ satisfy

$$k^i_A\partial_i\lambda_B - k^i_B\partial_i\lambda_A = -f_{AB}{}^C\lambda_C\,. \tag{5.26}$$

When $\lambda \neq 0$ all the objects of the theory with non-zero Kähler weight (in particular all the spinors and the symplectic section $\mathcal{V}$) will feel the effect of the symplectic transformation through a Kähler transformation. Infinitesimally one has

$$\delta_{\text{Kähler}}\mathcal{V} = -\frac{1}{2}(\lambda-\lambda^*)\mathcal{V}\,, \tag{5.27}$$

as follows from eq. (B.13). Next we introduce the momentum map, denoted by $\mathcal{P}^0_A$ and defined by

$$\mathcal{P}^0_A \equiv ik_A{}^i\partial_i\mathcal{K} - i\lambda_A\,. \tag{5.28}$$

One then readily shows that $\delta\mathcal{V}$, given via equations (5.22) and (5.4), can be written as

$$\delta\mathcal{V} = \alpha^A\left(k_A{}^i\mathcal{D}_i\mathcal{V} + i\mathcal{P}^0_A\mathcal{V} - \frac{1}{2}(\lambda_A-\lambda^*_A)\mathcal{V}\right)\,. \tag{5.29}$$

Since $\mathcal{V}$ only transforms under symplectic and Kähler transformations we conclude[5] that we must have

$$\delta\mathcal{V} = T\mathcal{V} - \frac{1}{2}(\lambda-\lambda^*)\mathcal{V}\,, \qquad \text{where} \qquad T\mathcal{V} = \alpha^A\left(k_A{}^i\mathcal{D}_i\mathcal{V} + i\mathcal{P}^0_A\mathcal{V}\right)\,, \tag{5.30}$$

where T is a generator of $\mathfrak{sp}(2n_V+2)$. Taking the product of the r.h.s. of the second equation with $\mathcal{V}$ we get the additional condition that the generators of G_V must satisfy:

$$\langle\,\mathcal{V}\mid T_A\mathcal{V}\,\rangle = 0\,. \tag{5.31}$$

The set of generators T_A which satisfy the constraint (5.31) and which form a subgroup of $\mathfrak{sp}(2n_V+2,\mathbb{R})$ is sometimes referred to as the *duality symmetry Lie algebra* [21]. Since, on the other hand

$$\delta\mathcal{V} = \pounds_{\alpha^A k_A}\mathcal{V} = \alpha^A\left(k_A{}^i\partial_i\mathcal{V} + k_A{}^{*\,i^*}\partial_{i^*}\mathcal{V}\right)\,, \tag{5.32}$$

we can write

$$\pounds_{\alpha^A k_A}\mathcal{V} - T\mathcal{V} + \frac{1}{2}(\lambda-\lambda^*)\mathcal{V} = 0\,, \tag{5.33}$$

[5] Actually, this is a consequence of requiring that the reparametrizations generated by the Killing vectors preserve not just the metric but the whole special Kähler geometry. This is what we are implicitly doing here and it is a condition necessary to have symmetries of the complete supergravity theory and not just of the bosonic equations of motion. We thank Patrick Meessen for a useful discussion on this point.

as the necessary and sufficient condition for the transformation to be a symmetry of the supergravity theory.[6]

One verifies that the above way of writing the action of T on $\mathcal{V}$, see eq. (5.30), satisfies eq. (5.7). By decomposing $T\mathcal{V}$ into the complete basis $\{\mathcal{V}, \mathfrak{D}_i\mathcal{V}, \mathcal{V}^*, \mathfrak{D}_{i^*}\mathcal{V}^*\}$ for the space of symplectic sections (see appendix B below eq. (B.9)) we find

$$\mathcal{P}^0_A = -\langle \mathcal{V} \mid T_A \mathcal{V}^* \rangle \,, \qquad \text{and} \qquad k_A{}^i = -i\mathcal{G}^{ij^*}\partial_{j^*}\mathcal{P}^0_A \,. \tag{5.34}$$

Substituting (5.30) into expression (5.24) we obtain a manifestly symplectic-invariant expression for the Noether current

$$J_{N\mu} = 2i\langle \mathfrak{D}_\mu \mathcal{V}^* \mid T\mathcal{V} \rangle + \text{c.c.} - 4\langle \star\mathcal{F}_{\mu\nu} \mid T\mathcal{A}^\nu \rangle \,. \tag{5.35}$$

5.2 Dualizing the Noether current

In form notation the conservation of the Noether current 1-form J_N is just $d\star J_N = 0$. We can define a 3-form[7] $G = \star J_N$, which satisfies $dG = 0$, so that locally $G = dB$. Note that G is not gauge invariant because J_N is not, either, due to the term $\hat{J}$ ($\delta_{\text{gauge}} G = \delta_{\text{gauge}} \hat{J}$). We can write this term in the form

$$\star\hat{J} = -4\langle \mathcal{F} \mid T\mathcal{A} \rangle \,, \tag{5.36}$$

where the exterior product between the forms in the symplectic inner product is always assumed and as a result the 2-form B gauge transformation is given by

$$\delta_{\text{gauge}} B = d\Lambda_1 - 4\langle \mathcal{F} \mid T\Lambda \rangle \,, \tag{5.37}$$

where the symplectic vector Λ is defined through eq. (3.4).

We can define the following gauge-invariant 2-form field strength

$$H = dB + 4\langle \mathcal{F} \mid T\mathcal{A} \rangle \,. \tag{5.38}$$

It is then clear that H is dual to the scalar part of the Noether current J_N,

$$H = \star(J_N - \hat{J}) \,. \tag{5.39}$$

The scalar part of the Noether current is proportional to the Killing vectors. At any given point there are only $2n_V$ (real) independent vectors. Thus, if we allow for Z^i-dependent coefficients, in general we will find linear combinations of scalar parts of the Noether currents. As a result, there will be as many constraints on the 2-form field strengths H_A and, at most there will be $2n_V$ independent real 2-forms.

[6]This condition can be read in two different ways: the Lie derivative of the section $\mathcal{V}$ has to vanish up to symplectic and Kähler transformations or the symplectic- and Kähler-covariant Lie derivative of $\mathcal{V}$ has to vanish identically.

[7]Of course, we have $\dim G_V$ Noether currents and as many dual 3-forms G_A but it is convenient to work with $G = \alpha^A G_A$.

5.3 The 2-form supersymmetry transformation

In the previous Subsection we have constructed a set of 2-forms associated to the isometries of the special Kähler manifold of ungauged $N=2, d=4$ supergravity and we have found their gauge transformations. Our goal in this section is to find their supersymmetry transformations. The main requirement that the proposed supersymmetry transformation of the 2-form B must satisfy is that the commutator agrees with the universal local supersymmetry algebra of the theory given in eq. (2.27) and which may be extended to include 2-forms to

$$[\delta_\eta, \delta_\epsilon] = \delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda) + \delta_{\text{gauge}}(\Lambda_1)\,. \tag{5.40}$$

The expressions for ξ and Λ are given by eqs. (2.28) and (3.5), respectively. The 2-form gauge transformation parameter Λ_1 is to be found in terms of η and ϵ.

Since B is defined by $dB = \star J_N$, the commutator of two supersymmetry variations on B must close into the algebra (5.40). We have

$$\delta_{\text{g.c.t.}}(\xi)B_{\mu\nu} = \pounds_\xi B_{\mu\nu} = \xi^\rho\partial_\rho B_{\mu\nu} + (\partial_\mu\xi^\rho)B_{\rho\nu} + (\partial_\nu\xi^\rho)B_{\mu\rho} = \xi^\rho(dB)_{\rho\mu\nu} - 2\partial_{[\mu}\left(\xi^\rho B_{\nu]\rho}\right)\,, \tag{5.41}$$

with $\pounds_\xi B_{\mu\nu}$ the Lie derivative of $B_{\mu\nu}$ with respect to ξ^ρ. Further, $\delta_{\text{gauge}}(\Lambda_1)B_{\mu\nu}$ is given in eq. (5.37). Hence, the supersymmetry transformations of $B_{\mu\nu}$ must lead to the commutator

$$[\delta_\eta, \delta_\epsilon]B_{\mu\nu} = \xi^\rho\frac{1}{\sqrt{|g|}}\epsilon_{\rho\mu\nu\sigma}J_N{}^\sigma - 4\langle\, \mathcal{F}_{\mu\nu} \mid T\Lambda\,\rangle + 2\partial_{[\mu}\left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho}\right)\,, \tag{5.42}$$

where we have substituted the duality relation, eq. (5.39), for $(dB)_{\mu\rho\sigma}$ in (5.41).

We make the following Ansatz for the supersymmetry transformation of $B_{\mu\nu}$ (up to lowest order in fermions),

$$\begin{aligned}\delta_\epsilon B_{\mu\nu} &= a\langle\, \mathfrak{D}_i\mathcal{V} \mid T\mathcal{V}^*\,\rangle\,\bar\epsilon_I\gamma_{\mu\nu}\lambda^{iI} + \text{c.c.}\\ &\quad + b\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle\,\bar\epsilon^I\gamma_{[\mu}\psi_{I\nu]} + \text{c.c.}\\ &\quad + c\langle\, \mathcal{A}_{[\mu} \mid T\delta_\epsilon\mathcal{A}_{\nu]}\,\rangle\,.\end{aligned} \tag{5.43}$$

This Ansatz is based on the requirement that all terms must have Kähler weight zero and that the 2-forms are real valued. The matrix T satisfies eq. (5.31).

We evaluate the commutator as follows. First we perform standard gamma matrix manipulations, change the order of the spinors, evaluate the complex conjugated terms and use relations from special geometry. Exhausting all such operations using formulae from appendices A and B leads to the following expression for the commutator

$$\begin{aligned}[\delta_\eta, \delta_\epsilon]B_{\mu\nu} &= 4ia\xi^\sigma\frac{1}{\sqrt{|g|}}\epsilon_{\sigma\mu\nu\rho}\left[\langle\, \mathfrak{D}^\rho\mathcal{V} \mid T\mathcal{V}^*\,\rangle - \langle\, \mathfrak{D}^\rho\mathcal{V}^* \mid T\mathcal{V}\,\rangle\right]\times\\ &\quad\times\big[+4ia\langle\, \mathfrak{D}_i\mathcal{V} \mid T\mathcal{V}^*\,\rangle\mathcal{G}^{ij^*}\langle\, \mathfrak{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}_{\mu\nu}\,\rangle\epsilon^{IJ}\bar\eta_I\epsilon_J\\ &\quad - 2b\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle\langle\, \mathcal{V}^* \mid \mathcal{F}_{\mu\nu}\,\rangle\epsilon^{IJ}\bar\eta_I\epsilon_J + \text{c.c.}\big]\\ &\quad - 8a\xi_{[\nu}\partial_{\mu]}\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle + 4ib\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle\partial_{[\mu}\xi_{\nu]} + c\langle\mathcal{A}_{[\mu} \mid [\delta_\eta,\delta_\epsilon]\mathcal{A}_{\nu]}\,\rangle\,,\end{aligned} \tag{5.44}$$

where it has been assumed that a and ib are real parameters. The parameter ξ^ρ is given by (2.28). The notation $[\cdots + \text{c.c.}]$ means that one should take the complex conjugate of whatever is written on the left within the brackets. The parameter a has been chosen to be real in order to obtain the scalar part of the Noether current in the first line of (5.44). The parameter ib has been chosen to be real so that the Kähler connection 1-form $\mathcal{Q}_\mu$ appearing in $\delta_\epsilon \Psi_{I\,\mu}$ cancels when adding the complex conjugated terms. We then take $2b = 4ia$ so that the first and the second term of the third line of eq. (5.44) combine into a 2-form gauge transformation parameter. Expression (5.44) is further manipulated using the completeness relation eq. (B.10). This is the step where we impose the condition that T must satisfy eq. (5.31). Using next the result for the 1-form commutator, eq. (3.2), to write out the term proportional to c in (5.44), we obtain

$$\begin{aligned}
[\delta_\eta, \delta_\epsilon] B_{\mu\nu} =& 4ia\xi^\sigma \frac{1}{\sqrt{|g|}} \epsilon_{\sigma\mu\nu\rho} \left[\langle\, \mathfrak{D}^\rho \mathcal{V} \mid T\mathcal{V}^*\,\rangle - \langle\, \mathfrak{D}^\rho \mathcal{V}^* \mid T\mathcal{V}\,\rangle\right] - 8a\partial_{[\mu}\left(\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle \xi_{\nu]}\right) \\
&+ 16a\langle\, \mathcal{F}_{\mu\nu} \mid T\,(\Lambda + \xi^\rho A_\rho)\,\rangle - \frac{c}{8}\xi^\sigma \frac{1}{\sqrt{|g|}} \epsilon_{\sigma\mu\nu\rho} \hat{J}^\rho - c\partial_{[\mu}\langle\, \mathcal{A}_{\nu]} \mid T\,(\Lambda + \xi^\rho \mathcal{A}_\rho)\rangle \\
&+ \frac{c}{2}\langle\, \mathcal{F}_{\mu\nu} \mid T\Lambda\,\rangle + c\langle \mathcal{F}_{\mu\nu} \mid T\xi^\rho \mathcal{A}_\rho\,\rangle\,,
\end{aligned} \tag{5.45}$$

where Λ is the 1-form gauge transformation parameter given in (3.5). This can be seen to be equal to the desired result, eq. (5.42), for $c = -16a$ and $a = -1/2$. We thus obtain the following supersymmetry variation rule for $B_{\mu\nu}$

$$\begin{aligned}
\delta_\epsilon B_{\mu\nu} =& -\frac{1}{2}\langle\, \mathfrak{D}_i \mathcal{V} \mid T\mathcal{V}^*\,\rangle\, \bar{\epsilon}_I \gamma_{\mu\nu} \lambda^{iI} + \text{c.c.} \\
& -i\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle\, \bar{\epsilon}^I \gamma_{[\mu} \psi_{I\nu]} + \text{c.c.} \\
& +8\langle\, \mathcal{A}_{[\mu} \mid T\delta_\epsilon \mathcal{A}_{\nu]}\,\rangle\,.
\end{aligned} \tag{5.46}$$

The 1-form gauge transformation parameter Λ_μ is given by

$$\Lambda_\mu = 2\langle\, \mathcal{V} \mid T\mathcal{V}^*\,\rangle \xi_\mu - 4\langle\, \mathcal{A}_\mu \mid T(\Lambda + \xi^\rho \mathcal{A}_\rho)\,\rangle + \xi^\rho B_{\mu\rho}\,. \tag{5.47}$$

6. World-sheet actions: the vector case

In this section we will construct the leading terms of the bosonic part of a κ-invariant world-sheet action for the stringy cosmic strings that couple to the 2-form potentials B that were constructed in section 5. Just as in the 0-brane case of section 4, we will construct actions which are manifestly symplectic invariant.

According to the results of the previous sections we expect to have strings which carry charges with respect to each of the $\dim G_V$ 2-forms $B_{A\,\mu\nu}$ that one can define. We define a $\dim G_V$-dimensional charge vector q^A. Symplectic invariance suggests a world-sheet action with leading terms

$$S = q^A \int d^2\sigma\, \langle \mathcal{V} \mid T_A \mathcal{V}^* \rangle \sqrt{|g_{(2)}|} + cq^A \int B_A\,, \tag{6.1}$$

where $g_{(2)}$ and B_A are the pullbacks of the space-time metric and 2-forms onto the world-sheet, respectively and where c is some normalization constant that will be fixed later. The tension of the string is given by the momentum map $\mathcal{P}^0_A$ as given in eq. (5.34).

Under supersymmetry the 2-form B_2 appearing in the Wess-Zumino term of Eq (6.1), transforms in part to 1-forms, see eq. (5.46). For the case of D-branes, one also encounters higher rank forms that transform non-trivially under gauge transformations of lower rank forms. In that case the structure of the Wess-Zumino term is constrained by the requirement of gauge invariance. This leads to the introduction of Born-Infeld vectors on the world-volume. However, in the present case it is impossible to make (6.1) gauge invariant under the gauge transformation (5.37) by adding additional terms to the Wess-Zumino term without adding more (scalar) degrees of freedom to the 2-dimensional world-sheet theory. If we restrict ourselves to backgrounds on which the 1-form field strengths are vanishing then the action (6.1) preserves half of the supersymmetries with the projector

$$\frac{1}{2}(1 + 4c\gamma_{01})\epsilon_I = 0 \quad \text{with} \quad c = \frac{1}{4}\,. \tag{6.2}$$

Actually, the same problem arises in the construction of a κ-symmetric world-sheet action for the heterotic superstring in backgrounds with non-trivial Yang-Mills fields since the NSNS 2-form transforms under Yang-Mills gauge transformations similar to eq. (5.37). In the 10-dimensional case of strings propagating on backgrounds with non-trivial Yang-Mills fields the solution to this puzzle lies in the addition of heterotic fermions to the world-sheet action whose gauge transformations cancel against the Yang-Mills part of the NSNS 2-form gauge transformation [22]. We suggest that a similar effect could be at work here. So the terms $\langle\, \mathcal{A}_{[\mu} \mid T\delta\mathcal{A}_{\nu]}\rangle$ in the 2-form supersymmetry transformation rule, eq. (5.46), and in the 2-form gauge transformation, eq. (5.37), should be canceled by anomalous terms in the supersymmetry transformations and gauge transformations of the world-sheet spinors.

We will see in the next section that the stringy cosmic string solutions for which the above action provides the sources require in order to preserve half of the supersymmetries exactly the same condition to be satisfied by the Killing spinor.

7. Supersymmetric vector strings

Stringy cosmic string solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets were found in [8].[8] They preserve half of the original supersymmetries and belong to the 'null class' of supersymmetric solutions characterized by the fact that the Killing vector that one can construct from their Killing spinors is null. Generically solutions in this class have Brinkmann-type metrics

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{-\mathcal{K}(Z,Z^*)}dzdz^*\,, \tag{7.1}$$

where $\mathcal{K}$ is the Kähler potential of the vector scalar manifold and where $\hat{\omega}$ is determined from the equation

$$(d\hat{\omega})_{\underline{zz^*}} = 2ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}\,, \tag{7.2}$$

[8]Solutions related to these by dimensional reduction have been obtained in a 3-dimensional context in ref. [23].

with $\mathcal{Q}_\mu$ the pullback of the Kähler 1-form connection given in eq. (B.2). The complex scalars Z^i are functions of u and z.

It is not easy to interpret physically these solutions for a generic dependence on the null coordinate u. When there is no dependence on u we can take $\hat{\omega} = 0$ and the metric is that of a superposition of cosmic strings (described by $\mathcal{K}$) lying in the direction $u - v$ and gravitational and electromagnetic waves (described by H) propagating along the same direction.

Setting $H = 0$ (which generically requires that we switch off all the electromagnetic fields) we obtain solutions that only describe cosmic strings. In order to study the behavior of these solutions under the symmetries of the theory, it is convenient to express them in an arbitrary system of holomorphic coordinates, which amounts to the introduction of an arbitrary holomorphic function $f(z)$ whose absolute value appears in the metric and whose phase appears in the Killing spinors of the solution

$$\begin{cases} ds^2 = 2dudv - 2e^{-\mathcal{K}(Z,Z^*)}|f|^2 dzdz^* \,, \\ \\ Z^i = Z^i(z)\,, \qquad f = f(z)\,, \\ \\ \epsilon_I = (f/f^*)^{1/4}\epsilon_{I\,0}\,, \qquad \gamma_{z^*}\epsilon_{I\,0} = 0\,. \end{cases} \tag{7.3}$$

If we take $z = x_2 + ix_3$ then the condition $\gamma_{z^*}\epsilon_{I\,0} = 0$ is equivalent to eq. (6.2).

The holomorphic functions $Z^i(z), f(z)$ are assumed to be defined on the Riemann sphere $\hat{\mathbb{C}}$, but, generically, they will not be single-valued on it due to the presence of branch cuts. These branch cuts are to be associated with the presence of cosmic strings just as was done in the particular case of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ special Kähler manifold studied in refs. [2] and [5].

As a general rule bosonic fields must be single-valued unless they are subject to a gauge symmetry which forces us to identify as physically equivalent those configurations which are related by admissible gauge transformations. In the theories that we are considering the complex scalars $Z^i(z)$ do not transform under any gauge symmetry. Only the global group of isometries G_V of $\mathcal{G}_{ij^*}$ acts on them and only a discrete subgroup $G_V(\mathbb{Z}) \subseteq \mathrm{Sp}(2n_V+2,\mathbb{Z})$ will be a global symmetry at the quantum level.

In the resulting theories two values of $Z^i(z)$ may be considered equivalent if they are related by a $G_V(\mathbb{Z})$ transformation. This enables one to construct solutions in which the scalars $Z^i(z)$ are multi-valued functions with branch cuts related to the elements of $G_V(\mathbb{Z})$. The source for a branch cut is provided by the Wess-Zumino term of a cosmic string. This is explained in detail for the 10-dimensional case of the 7-branes in [2].

Next we discuss the emergence of axions related to the presence of Killing vectors. For every Killing vector $\alpha^A k_A{}^i$ one can always find an adapted coordinate system $\{Z^i\}$ such that the metric $\mathcal{G}_{ij^*}$ does not depend on the real part of the coordinate Z^1, say. In this coordinate system $\alpha^A k_A{}^i \partial_i = \partial_1$ and the isometries generated by it act as constant shifts of Z^1 by a real constant:

$$\delta Z^1 = c \in \mathbb{R}\,. \tag{7.4}$$

This transformation only acts on the real part of Z^1, χ^1, which is, then, what it is sometimes meant by an axion: a real scalar field with no non-derivative couplings to the other scalars and with a shift symmetry.[9]

It is clear that we can, in principle, define as many different axion fields as there are independent Killing vectors,[10] i.e. $\dim G_V$, i.e. as many as 2-forms, which can be understood as their duals. Their (both those of the axions and 2-forms) equations of motion are not necessarily independent, though, and they will satisfy a number of constraints, as discussed before, and, at most, there can be $2n_V$ independent axions.

We now discuss the properties of the cosmic string solutions in a local neighborhood of the location z_0 in the transverse space of a cosmic string. Infinitesimally the transformation of the scalars Z^i when going around z_0 is given by eq. (5.4). In some coordinate basis, the transformation will only be an axion shift.

Besides the scalars Z^i also the Killing spinors ϵ_I will undergo transformations when going around the cosmic string at z_0. This is because when the scalars transform as in eq. (5.4) the Kähler potential transforms as

$$\mathcal{K}(Z', Z'^*) = \mathcal{K}(Z, Z^*) + \lambda_\alpha(Z) + \lambda^*_\alpha(Z^*)\,. \tag{7.5}$$

From the fact that the Killing spinor ϵ_I has Kähler weight 1/2 it then follows that

$$\epsilon_I(z) \to e^{\frac{1}{4}[\lambda_\alpha - \lambda^*_\alpha] + \frac{i}{2}\varphi_\alpha}\epsilon_I(z)\,, \tag{7.6}$$

when going around z_0. The phases φ_α relate to the fact that in general the spinors transform under the double cover of G_V.[11] The Killing spinor ϵ_I is defined in terms of the holomorphic function $f(z)$ via eqs. (7.3). The monodromy of f when going around z_0 must be

$$f(z) \to e^{\lambda_\alpha[Z(z)] + i\varphi_\alpha} f(z)\,. \tag{7.7}$$

The cosmic string solutions contain information about the moduli space of the theory, i.e. the space of inequivalent values for Z^i. The classical moduli space is defined by the requirement

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} < 0\,, \tag{7.8}$$

[9] A more precise definition would require χ^1 to be a pseudoscalar too. Actually, the real and imaginary parts of the complex scalars in $N = 2, d = 4$ vector supermultiplets have different parities, but, in a general model with arbitrary coordinates one should look at the couplings to the vector fields to determine the parity of χ^1.

On the other hand, the action of $N = 2, d = 4$ supergravity indicates that the axions must appear in $\Re\mathrm{e}\,\mathcal{N}_{\Lambda\Sigma}$, which couples to the parity-odd term $F^\Lambda \wedge F^\Sigma$. Under symplectic transformations $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ $\Re\mathrm{e}\,\mathcal{N}$ is shifted to $\Re\mathrm{e}\,\mathcal{N} + C$, as one expects from axions. This suggests another possible characterization of axions: χ^1 is an axion if its shifts are embedded in the Abelian subgroup of symplectic transformations of the form $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$.

[10] However, they cannot be used simultaneously, since we can only use simultaneously adapted coordinates for commuting isometries.

[11] One can even include yet another phase factor in the transformation rule for the Killing spinors which incorporates the fact that ϵ_I may come back to itself up to a sign, i.e. one can include nontrivial spin structures.

in order that the kinetic terms of the 1-forms have the right sign in the action (2.1). The zeros of the polynomial $\delta Z^i = \alpha^A k_A{}^i$ which belong to the space (7.8) (or possibly on the boundary thereof) are fixed points of the monodromy and therefore comprise the loci of the cosmic strings in the quantum moduli space:

$$\{Z^i \,|\, \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} < 0\}/\mathrm{G_V}(\mathbb{Z})\,. \tag{7.9}$$

Drawing from the analogy with the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ case studied in [5] one can expect all physical properties of globally well-defined stringy cosmic string solutions to be mapped into geometrical properties of the space (7.9). Such properties are the total mass, possible deficit angles at the sites of the cosmic strings, orders of monodromy transformations (the number of times the same monodromy has to be applied in order to equal the identity), etc. Here we will not attempt to work out the global properties of these solutions, since they are strongly model-dependent.

In the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ case one could have derived all geometrical properties of the quantum moduli space $\mathrm{SL}(2,\mathbb{Z})\backslash\,\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ by studying the globally well-defined supersymmetric stringy cosmic string solutions. It is therefore natural to ask the question whether this is generally true, i.e. whether (some class of) quantum moduli spaces of Calabi-Yau reduced supergravities can be obtained by studying the properties of the stringy cosmic string solutions.

We leave this for a future investigation.

8. The 2-forms: the hyper case

If we consider $N = 2$, $d = 4$ supergravity with general matter couplings, we can have apart from the complex scalars in the vector multiplets $4n_H$ real scalars when coupling gravity to n_H hypermultiplets. In the following we repeat the program of introducing 2-forms in order to dualize the hyperscalars which parameterize the Noether currents of some isometry group of the quaternionic Kähler manifold. We first construct the Noether currents, dualize them and subsequently construct the supersymmetry transformation rule for the dual 2-forms. For the subset of commuting isometries a similar program has been performed in [24] where also actions for the dualized scalars are given.

8.1 The Noether current

The transformations we are dealing with are just the isometries of the quaternionic Kähler manifold that we write in the form

$$\delta q^u = \alpha^A k_A{}^u(q)\,, \tag{8.1}$$

where $k_A{}^u$ are the components of the Killing vectors $k_A = k_A{}^u\partial_u$ that generate the isometry group G_H of H_{uv}. The parameters α^A are real parameters.

Associated to each of the isometries we can define a momentum map[12] $\mathsf{P}_{AI}{}^{J}$ defined by the equation

$$\mathfrak{D}_u \mathsf{P}_{AI}{}^{J} = -\mathsf{J}_I{}^J{}_{uv} k_A{}^v \,, \tag{8.2}$$

where $\mathsf{J}_I{}^J{}_{uv}$ is the triplet complex structures of the quaternionic-Käher manifold.

We write the triplet of complex structures $\mathsf{J}_I{}^J{}_{uv}$ in terms of the Quadbeins as follows

$$\mathsf{J}_I{}^J{}_{uv} = \frac{i}{2}(\sigma_x)_I{}^J\, \mathsf{J}^x{}_{uv} \quad \text{with} \quad \mathsf{J}^{x\,u}{}_v = -i\mathsf{U}^{\alpha I}{}_v(\sigma_x)_I{}^J \mathsf{U}_{\alpha J}{}^u \,, \tag{8.3}$$

where the σ_x, $x = 1,2,3$, are the three Pauli matrices. We will often write $\mathsf{P}_I{}^J \equiv \alpha^A \mathsf{P}_{AI}{}^J$.

The Noether current associated to the these isometries, which do not act on the vector fields, is just

$$J^\mu_N = \delta q^u \frac{1}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu q^u)} = 4\mathsf{H}_{uv}\partial^\mu q^v \delta q^u \,, \tag{8.4}$$

and satisfies $\nabla_\mu J^\mu_N = 0$.

8.2 Dualizing the Noether current

Since the isometries of the quaternionic Kähler manifold do not act on the vectors of the theory they are symmetries of the action and there will be no anomalous contribution to the Noether current such as $\hat{J}$ which we encountered when discussing the isometries of the special Kähler manifold. We can thus immediately define the gauge-invariant 3-form field strength H via

$$H = dB = \star J_N \,, \tag{8.5}$$

where $H = \alpha^A H_A$ and $B = \alpha^A B_A$.

8.3 The 2-form supersymmetry transformation

We know that, since B is defined by $dB = \star J_N$, the commutator of two supersymmetry variations on B must close into the algebra (5.40), i.e. it must lead to the commutator

$$[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = \xi^\rho \frac{1}{\sqrt{|g|}} \epsilon_{\rho\mu\nu\sigma} J_N{}^\sigma + 2\partial_{[\mu}\left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho}\right) \,. \tag{8.6}$$

In order to achieve this, we make the following Ansatz for the supersymmetry variation of the 2-form (up to lowest order in fermions)

$$\begin{aligned} \delta_\epsilon B_{\mu\nu} = {} & a\mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{[\mu}\psi_{J|\nu]} + \text{c.c.} \\ & + b\mathsf{U}_{\alpha J}{}^u \mathfrak{D}_u \mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{\mu\nu}\zeta^\alpha + \text{c.c.} \,, \end{aligned} \tag{8.7}$$

where a and b are arbitrary complex constants.

[12] Momentum maps play a crucial role in the gauging of the isometries. It is therefore interesting to note that the mathematics which governs the 2-forms is similar to that used in gauged matter coupled $N = 2$, $d = 4$ supergravity.

Evaluating the commutator and assuming that a and ib are real parameters we obtain

$$\begin{aligned}
[\delta_\eta, \delta_\epsilon] B_{\mu\nu} &= -\frac{3}{2} ib(\star dq^w)_{\mu\nu\rho}\xi^\rho \mathsf{H}_{vw}\delta q^v \\
&+\frac{3}{2} ib \mathsf{J}_I{}^K{}_{vw}\delta q^v \partial_{[\nu} q^w X_{\mu]K}{}^I \\
&+2\partial_{[\mu}\left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho}\right) - a\mathsf{J}_I{}^K{}_{vw}\delta q^v \partial_{[\nu} q^w X_{\mu]K}{}^I \,,
\end{aligned} \tag{8.8}$$

where we have defined the matrix of vector fields

$$X_{\mu I}{}^J \equiv -\bar{\eta}^J \gamma_\mu \epsilon_I - \bar{\eta}_I \gamma_\mu \epsilon^J \,, \tag{8.9}$$

and where the gauge parameter Λ_μ is given by

$$\Lambda_\mu = -\frac{a}{2} X_J{}^I{}_\mu \mathsf{P}_I{}^J + \xi^\rho B_{\mu\rho}. \tag{8.10}$$

Next we choose $a = \frac{3}{2} ib$ and we are left with

$$[\delta_\eta, \delta_\epsilon] B_{\mu\nu} = -\frac{3}{2} ib(\star dq^w)_{\mu\nu\rho}\xi^\rho \mathsf{H}_{vw}\delta q^v + 2\partial_{[\mu}\left(\Lambda_{\nu]} - \xi^\rho B_{\nu]\rho}\right) \,. \tag{8.11}$$

If we compare this expression with eq. (8.6) using eq. (8.4) we read off that $ib = -\frac{8}{3}$, so that $a = -4$.

The supersymmetry transformation of the 2-forms dual to the hyperscalars parameterizing the Noether current (8.4) is thus

$$\begin{aligned}
\delta_\epsilon B_{\mu\nu} &= -4\mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{[\mu}\psi_{J|\nu]} + \text{c.c.} \\
&\quad +\frac{8i}{3} \mathsf{U}_{\alpha J}{}^u \mathfrak{D}_u \mathsf{P}_I{}^J \bar{\epsilon}^I \gamma_{\mu\nu}\zeta^\alpha + \text{c.c.} \,,
\end{aligned} \tag{8.12}$$

and the 2-form gauge parameter Λ_μ is given by

$$\Lambda_\mu = 2X_J{}^I{}_\mu \mathsf{P}_I{}^J + \xi^\rho B_{\mu\rho}. \tag{8.13}$$

9. World-sheet actions: the hyper case

Stringy cosmic strings in the hyper case are strings electrically charged under the 2-forms B constructed in section 8. In this section we will construct the bosonic part of the string effective action, which preserves half of the supersymmetries of the theory. In analogy with the Ansatz that we made for the strings in the vector case we again express the tension of the string in terms of the momentum maps. We make the following Ansatz

$$S = \int d^2\sigma \mathcal{T}_1 \sqrt{|g_{(2)}|} + c\, q^A \int B_A, \tag{9.1}$$

where c is some real number which will be fixed later. The tension is given by

$$\mathcal{T}_1 = \sqrt{(\mathsf{P}^x)^2} \quad \text{where} \quad \mathsf{P}^x = \alpha^A \mathsf{P}^x{}_A \quad \text{with} \quad \mathsf{P}_I{}^J = \frac{i}{2}\mathsf{P}^x(\sigma_x)_I{}^J \tag{9.2}$$

and in taking the square we sum over $x = 1, 2, 3$.

Performing a supersymmetry variation of the action (9.1) using the transformation rules (2.16), (2.19) and (8.12) we find that the string action preserves half of the supersymmetries with a projector given by

$$\Pi_I{}^J = \frac{1}{2}(\delta_I{}^J - \frac{8ci}{\sqrt{(\mathsf{P}^x)^2}}\mathsf{P}_I{}^J\gamma_{01}), \qquad \Pi_I{}^J\epsilon^I = 0, \quad \text{where } c = -\frac{1}{4}. \tag{9.3}$$

An important distinction with the analogous string action constructed in section 6 is that in the present case the Wess-Zumino term is gauge invariant up to a total derivative whereas in the case of strings coupled to 2-forms dual to vector scalars the Wess-Zumino term is not by itself gauge invariant, cf. the discussion below eq. (6.1). In fact one may consider the action (9.1) as the first example of a 1/2 BPS $(d-3)$-brane action which is well-defined (at the bosonic level) for all possible $(d-2)$-form potentials. In the $d = 10$-dimensional situation only the brane actions related to the D7-branes are well understood. For the other 8-forms which couple to the Q7-branes of [2] there are still open problems regarding a proper understanding of the world-volume dynamics. The fact that in the particular case of the hyperstrings we can construct well-defined actions supports the idea that in general one can treat all isometries of any scalar sigma model in any supergravity on an equal footing (provided they pertain to be discrete isometries of the quantum moduli space). This suggests that in order to find the full spectrum of 1/2 BPS states one best considers the same supergravity theory in various coordinate systems in which these isometries take on a simple form.

10. Supersymmetric hyperstrings

In ref. [25] it was shown that the c-map transforms supersymmetric stringy cosmic string solutions of the vector scalar manifold into supersymmetric stringy cosmic string solutions of the hyperscalar manifold. The latter belong to the timelike class of supersymmetric solutions characterized by the fact that the Killing vector that one can construct from the Killing spinors of the solution is timelike. The metric for this class of solutions (for vanishing vector multiplets) takes the following form

$$ds^2 = dt^2 - \gamma_{\underline{mn}}dx^m dx^n\,. \tag{10.1}$$

The 3-dimensional spatial metric $\gamma_{\underline{mn}}$ (or its Dreibeins $V^x{}_{\underline{m}}$) is related to the hyperscalars $q^u(x)$ by two conditions. The first condition is

$$V_x{}^{\underline{m}}\,\partial_{\underline{m}}q^u\;\mathsf{U}^{\alpha J}{}_u\,(\sigma_x)_J{}^I \;=\; 0\,, \tag{10.2}$$

and the second condition reads, in a given SU(2) and Lorentz gauge,

$$\varpi_{\underline{m}}{}^{xy} = \varepsilon^{xyz}\mathsf{A}^z{}_u\;\partial_{\underline{m}}q^u\,, \tag{10.3}$$

where $\varpi_{\underline{m}}{}^{xy}$ is the spin connection 1-form of the 3-dimensional metric and $\mathsf{A}^z{}_u\partial_{\underline{m}}q^u$ is the pullback of the SU(2) connection of the quaternionic-Kähler manifold parameterized by the scalars q^u. In the gauge in which eq. (10.3) holds the Killing spinors take the form

$$\epsilon_I = \epsilon_{I\,0}, \quad \Pi^x{}_I{}^J\ \epsilon_{J\,0} = 0 \quad \text{with} \quad \Pi^x{}_I{}^J \ \equiv\ \frac{1}{2}[\ \delta_I{}^J\ -\ \gamma^{0(x)}\ (\sigma_{(x)})_I{}^J\ ] \tag{10.4}$$

where the notation (x) in (10.4) means that x is not summed over so the constraints are imposed for each non-vanishing component of the SU(2) connection.

We now repeat for the hyperscalars parameterizing a quaternionic Kähler manifold with isometry group G_H the discussion of section 7. The fields will only depend on two spatial coordinates (x^1 and x^2, say, that can always be combined into a complex coordinate z) which parameterize the transverse space of the cosmic string. The metric will take the form

$$ds^2 = dt^2 - (dx^3)^2 - 2e^{\Phi(z,z^*)}dzdz^*\,, \tag{10.5}$$

and the hyperscalars will be real functions $q^u(z,z^*)$. A convenient Dreibein basis is

$$\hat{V}^3 = dx^3\,, \quad \hat{V}^z = V dz\,, \quad \hat{V}^{z^*} = V^* dz^*\,, \quad |V|^2 = e^{\Phi(z,z^*)}\,. \tag{10.6}$$

In this Dreibein basis the supersymmetry conditions eqs. (10.2) and (10.3) take the respective form

$$\mathsf{U}^{\alpha 2}{}_u\partial_{\underline{z}}q^u = \mathsf{U}^{\alpha 1}{}_u\partial_{\underline{z}^*}q^u\ = 0\,, \tag{10.7}$$

$$\varpi_{\underline{z}}{}^{zz^*}\ = \mathsf{A}^3{}_u\ \partial_{\underline{z}}q^u\,, \tag{10.8}$$

$$\mathsf{A}^1{}_u\ \partial_{\underline{m}}q^u = \mathsf{A}^2{}_u\ \partial_{\underline{m}}q^u\ = 0\,. \tag{10.9}$$

The Killing spinors of these solutions, in this basis, are given by

$$\epsilon_I = \epsilon_{I\,0}\,, \quad \Pi^3{}_I{}^J\ \epsilon_{J\,0} = 0\,. \tag{10.10}$$

It can be shown that in this gauge the pullbacks of the complex structures J^1 and J^2 vanish while J^3 remains nonzero and one recovers the projection operator eq. (9.3). As in the case of the vector scalars, it is convenient to work in a more general coordinate system in which the metric takes the form

$$ds^2 = dt^2 - (dx^3)^2 - 2e^{\Phi(z,z^*)}|f|^2dzdz^*\,, \tag{10.11}$$

where $f(z)$ is a holomorphic function. The supersymmetry conditions, eqs. (10.7) and (10.9), do not change and eq. (10.8) is still satisfied *with the old spin connection.* If the new spin connection is computed with respect to the new frame

$$\hat{V}^3 = dx^3\,, \quad \hat{V}^z = V f^* dz\,, \quad \hat{V}^{z^*} = V^* f dz^*\,, \tag{10.12}$$

then, we find that

$$\varpi_{\underline{z}}{}^{zz^*} = \varpi_{\underline{z}}{}^{zz^*}{}_{\text{old}} + \partial_{\underline{z}}\log f\,, \tag{10.13}$$

and then the Killing spinors take the form

$$\epsilon_I = e^{\frac{1}{2}\log(f/f^*)\gamma^{03}}\epsilon_{I\,0}\,, \tag{10.14}$$

the constant spinor $\epsilon_{I\,0}$ obeying the same constraints as above, eqs. (10.10). These same constraints allow us to rewrite it in the equivalent form

$$\epsilon_I = \exp\{\frac{1}{2}\log(f/f^*)\sigma_3\}_I{}^J \epsilon_{J\,0}\,. \tag{10.15}$$

The multi-valuedness of the Killing spinors ϵ_I of these solutions is related to the $\mathrm{U}(1) \subset \mathrm{SU}(2)$ gauge transformation where the $\mathrm{U}(1)$ subgroup is associated to the non-vanishing component $\mathsf{A}^3{}_u \partial_{\underline{z}} q^u$ of the SU(2) connection pulled back on the space-time. The transformations of the Killing spinors determine the monodromy properties of the holomorphic function f similarly to what happens in the case of the vector scalars.

11. Conclusions

In this paper we have shown how, consistent with the supersymmetry algebra, the standard set of bosonic fields of $N = 2, d = 4$ supergravity coupled to vector and hypermultiplets can be extended to include $n_V + 1$ additional "magnetic" vector fields and $\dim G_V$ 2-form fields dual to vector multiplet scalars, as well as $\dim G_H$ 2-form fields dual to hypermultiplet scalars. These fields couple, respectively, to magnetic 0-branes (black holes) and cosmic strings for which there are well-known classical solutions that we have reviewed. They are necessary to construct κ-symmetric effective world-volume actions for these solutions. We have studied the construction of these actions in a symplectic-covariant form and checked that their supersymmetry to lowest order precisely leads to the 1/2 BPS condition one expects for these solutions. The vector string action is 1/2 BPS on backgrounds of vanishing 1-form field strengths because the WZ term is not invariant under the 1-form gauge transformations. This problem is analogous to that of the gauge invariance of the heterotic string on a background of Yang-Mills fields. We propose that it may be solved in the same fashion [22], thanks to a cancelation with anomalous gauge transformations of fermions, which we are not considering at this order of approximation.

One possible extension is based on the idea that there may also be 3- and 4-form potentials (also known as deformation potentials and top-form potentials, respectively) unrelated by duality to any of the standard fields of the theory and which do not carry any (continuous) degree of freedom. Deformation and top-form potentials have been found and studied in 10-dimensional supergravities [26 – 29]. These potentials can be associated with higher-dimensional objects such as domain-walls and space-time-filling branes. For a recent derivation of the representations of these potentials for maximal supergravity from a Kac-Moody point of view, see [30, 31]. It would be very interesting to carry out a similar analysis in the $N = 2, d = 4$ theories. For the cases that the special Kähler manifold corresponds to a coset geometry the representations of these potentials again follow from a Kac-Moody approach [32]. Alternatively, some of the deformation and top-form potentials should be related by dimensional reduction to those of minimal $d = 5$ supergravity, which have recently been constructed in [33].[13] Further, the deformation potentials carry a great

[13] All the deformation and top-form potentials of minimal $d = 5$ supergravity will give rise to top-form potentials in 4 dimensions. However, in general, not all these potentials can be obtained from a higher-dimensional theory, the best-known example being the RR 9-form potential of $N = 2A, d = 10$ supergravity.

deal of information about possible gaugings or massive deformations (hence the name) of the supergravity theory. It would be interesting to work these things out in detail for the $N = 2, d = 4$ theories.

There is yet another interesting connection between gauged supergravity and the $(d-2)$-form potentials that we have studied here which is worth exploring. It is known that if one performs generalized (Scherk-Schwarz) dimensional reductions associated to one isometry of a sigma model metric in d space-time dimensions, one gets gauged supergravities [34−40] in $d-1$ space-time dimensions. Locally, these generalized dimensional reductions can be interpreted as reductions in the background of the $(d-3)$ brane that would couple to the $(d-2)$-form potential dual to the Noether current associated to the isometry used in the reduction [3, 40, 41]. After reduction, in the transverse direction, the $(d-3)$ branes become domain-wall solutions in the reduced theory and should couple to deformation potentials directly obtainable from the $(d-2)$-form potentials of the original theory.

In particular, in the case at hand, we should be able to perform explicit generalized dimensional reductions using isometries of the special Kähler manifold in a way consistent with all the symmetries of the theory (as it was done in [3]) down to 3 dimensions, obtaining gauged 3-dimensional supergravities on the one hand. On the other hand, we should be able to relate the deformation parameters that appear in 3 dimensions with deformation potentials (i.e. 2-form potentials) which can be obtained from the 4-dimensional 2-form potentials that we have obtained here. At the same time one should be able to relate the 4-dimensional cosmic string solutions to the 3-dimensional domain-wall solutions. Similar relations between the 5- and 4-dimensional theories must exist. Work on these subjects is in progress.

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A. Conventions

The signature is mostly minus. Flat tangent space indices are denoted by lower case Latin indices a whose values are $a = 0, 1, 2, 3$. Curved space-time indices are denoted by lower case Greek indices μ whose values are $\mu = t, \underline{1}, \underline{2}, \underline{3}$. The tangent space Levi-Cività symbol is taken to be $\epsilon^{0123} = -\epsilon_{0123} = 1$. The curved Levi-Cività tensor whose indices are lowered with the metric is taken to be

$$\epsilon^{\mu_1 \cdots \mu_4} = \sqrt{|g|} e^{\mu_1}_{a_1} \cdots e^{\mu_4}_{a_4} \epsilon^{a_1 \cdots a_4} \,, \tag{A.1}$$

where e^μ_a is the inverse Vielbein. The Hodge dual of a k-form ω is defined to be

$$(*\omega)_{\mu_1 \cdots \mu_{d-k}} = \frac{1}{k! \sqrt{|g|}} \epsilon_{\mu_1 \cdots \mu_{d-k} \nu_1 \cdots \nu_k} \omega^{\nu_1 \cdots \nu_k} \,. \tag{A.2}$$

The Riemann tensor is defined by $R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma^\sigma_{\nu\rho} + \cdots$.

We work in the Majorana representation which in signature $(+ - --)$ has all the gamma matrices purely imaginary,

$$\gamma_a^* = -\gamma_a \,. \tag{A.3}$$

The anticommutator is

$$\{\gamma_a, \gamma_b\} = +2\eta_{ab} \,. \tag{A.4}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d \,. \tag{A.5}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1 \cdots a_n} = \frac{(-1)^{[n/2]} i}{(4-n)!} \epsilon^{a_1 \cdots a_n b_1 \cdots b_{4-n}} \gamma_{b_1 \cdots b_{4-n}} \gamma_5 \,, \tag{A.6}$$

where $[n/2]$ is the highest integer less than or equal to $n/2$. The following two gamma matrix identities are used in the text

$$\gamma_{\mu\nu}\gamma_\rho = \gamma_{\mu\nu\rho} + \gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho} \,, \tag{A.7}$$

$$\gamma_{\mu\nu}\gamma_{\rho\sigma} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5 - 2g_{\mu[\rho}g_{\sigma]\nu} - 2\gamma_{\mu[\rho}g_{\sigma]\nu} + 2g_{\mu[\rho}\gamma_{\sigma]\nu} \,. \tag{A.8}$$

We use 4-component chiral spinors χ whose chirality is related to the position of the SU(2) index I or the position of the Sp($2n_H$) index α,

$$\gamma_5 \chi_I = -\chi_I \,, \quad \gamma_5 \chi^I = \chi^I \,, \tag{A.9}$$

$$\gamma_5 \chi_\alpha = -\chi_\alpha \,, \quad \gamma_5 \chi^\alpha = \chi^\alpha \,. \tag{A.10}$$

The position of the SU(2) index I and of the Sp($2n_H$) index α is raised and lowered under complex conjugation

$$\chi_I^* = \chi^I \quad \text{and} \quad \chi_\alpha^* = \chi^\alpha \,. \tag{A.11}$$

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The conjugated spinor is taken to be

$$\bar{\chi}_I = i(\chi^I)^\dagger\gamma_0 \quad \text{and} \quad \bar{\chi}_\alpha = i(\chi^\alpha)^\dagger\gamma_0 \,. \tag{A.12}$$

The spinors are anticommuting and we take the convention that they do not change their order under complex conjugation. We have the following property for spinor bilinears

$$\bar{\chi}_1\gamma^{\mu_1\cdots\mu_n}\chi_2 = (-1)^{[(n+1)/2]}\bar{\chi}_2\gamma^{\mu_1\cdots\mu_n}\chi_1 \,, \tag{A.13}$$

where χ_1 and χ_2 are arbitrary spinors.

B. Special Kähler geometry

A Kähler manifold $\mathcal{M}$ is a complex manifold with coordinates Z^i and $(Z^i)^* = Z^{*\,i^*}$ whose Kähler 2-form $\mathcal{J}$ is closed. The Kähler 2-form is then locally given by $\mathcal{J} = d\mathcal{Q}$ with $\mathcal{Q}$ the Kähler connection 1-form. Both the metric and the Kähler connection 1-form can be expressed in terms of the Kähler potential $\mathcal{K}$ as follows

$$ds^2 = 2\mathcal{G}_{ii^*}\, dZ^i dZ^{*\,i^*} \quad \text{with} \quad \mathcal{G}_{ii^*} = \partial_i\partial_{i^*}\mathcal{K}\,, \tag{B.1}$$

$$\mathcal{Q} \equiv (2i)^{-1}(dZ^i\partial_i\mathcal{K} - dZ^{*\,i^*}\partial_{i^*}\mathcal{K})\,. \tag{B.2}$$

The non-vanishing components of the Levi-Cività connection on a Kähler manifold are given by

$$\Gamma_{jk}{}^i = \mathcal{G}^{ii^*}\partial_j\mathcal{G}_{i^*k}\,, \qquad \Gamma_{j^*k^*}{}^{i^*} = \mathcal{G}^{i^*i}\partial_{j^*}\mathcal{G}_{k^*i}\,. \tag{B.3}$$

The Kähler potential is not unique. It is defined up to Kähler transformations,

$$\mathcal{K}(Z,Z^*) \to \mathcal{K}(Z,Z^*) + \lambda(Z) + \lambda^*(Z^*)\,, \tag{B.4}$$

where λ is any holomorphic function of the complex coordinates Z^i.

An object X is said to have Kähler weight q when X transforms under the above Kähler transformations as

$$X \to e^{-(q\lambda - q\lambda^*)/2}X\,. \tag{B.5}$$

The Kähler-covariant derivative $\mathfrak{D}$ acting on X has the following holomorphic and anti-holomorphic components

$$\mathfrak{D}_i X \equiv (\nabla_i + iq\mathcal{Q}_i)\,X\,, \qquad \mathfrak{D}_{i^*}X \equiv (\nabla_{i^*} - i\bar{q}\mathcal{Q}_{i^*})\,X\,, \tag{B.6}$$

where ∇ is the standard covariant derivative associated to the Levi-Cività connection, eqs. (B.3), on $\mathcal{M}$. For objects with Kähler weight q the space-time pullback of the Kähler-covariant derivative is given by

$$\mathfrak{D}_\mu = \nabla_\mu + iq\mathcal{Q}_\mu\,, \tag{B.7}$$

where ∇_μ is the standard space-time covariant derivative plus the pullback of the Levi-Cività connection on $\mathcal{M}$ if necessary and where $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form of eq. (B.2).

A special Kähler manifold is the base manifold of a $\mathrm{Sp}(2n_V+2,\mathbb{R})\times \mathrm{U}(1)$ bundle [11]. There exist sections $\mathcal{V}$ such that

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \quad \rightarrow \quad \begin{cases} \langle \mathcal{V} \mid \mathcal{V}^* \rangle & \equiv \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda = -i\,, \\ \mathfrak{D}_{i^*}\mathcal{V} & = (\partial_{i^*} - \tfrac{1}{2}\partial_{i^*}\mathcal{K})\mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V}\rangle = 0\,, \end{cases} \tag{B.8}$$

where $\mathfrak{D}_i\mathcal{V} = (\partial_i + \frac{1}{2}\partial_i\mathcal{K})\mathcal{V}$.

It follows from the basic definitions, eqs. (B.8), that

$$\begin{aligned} \mathfrak{D}_{i^*}\,\mathfrak{D}_i\mathcal{V} &= \mathcal{G}_{ii^*}\,\mathcal{V}\,, & \langle \mathfrak{D}_i\mathcal{V} \mid \mathfrak{D}_{i^*}\mathcal{V}^*\rangle &= i\,\mathcal{G}_{ii^*}\,, \\ \langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V}^*\rangle &= 0\,, & \langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V}\rangle &= 0\,, \\ \langle \mathfrak{D}_i\mathfrak{D}_j\mathcal{V} \mid \mathcal{V}\rangle &= 0\,, & \langle \mathfrak{D}_j\mathcal{V} \mid \mathfrak{D}_i\mathcal{V}\rangle &= 0\,. \end{aligned} \tag{B.9}$$

If we now group together $\mathcal{V}$ and $\mathfrak{D}_i\mathcal{V}$ into $\mathcal{E}_\Lambda = (\mathcal{V}, \mathfrak{D}_i\mathcal{V})$ we can see that $\langle \mathcal{E}_\Sigma \mid \mathcal{E}^*{}_\Lambda\rangle$ is a non-degenerate matrix. Using $\{\mathcal{E}_\Sigma, \mathcal{E}^*{}_\Lambda\}$ as a basis for the space of symplectic sections we obtain the following completeness relation

$$i\mathbb{1} = -\mid \mathcal{V}^*\rangle\langle \mathcal{V}\mid + \mid \mathcal{V}\rangle\langle \mathcal{V}^*\mid - \mathcal{G}^{ii^*}\mid \mathfrak{D}_i\mathcal{V}\rangle\langle \mathfrak{D}_{i^*}\mathcal{V}^*\mid + \mathcal{G}^{ii^*}\mid \mathfrak{D}_{i^*}\mathcal{V}^*\rangle\langle \mathfrak{D}_i\mathcal{V}\mid\,. \tag{B.10}$$

We write for the components of $\mathfrak{D}_i\mathcal{V}$ the following

$$\mathfrak{D}_i\mathcal{V} = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Sigma\,i} \end{pmatrix}. \tag{B.11}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is defined by the following two relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \qquad h_{\Lambda\,i} = \mathcal{N}^*{}_{\Lambda\Sigma} f^\Sigma{}_i\,. \tag{B.12}$$

The identity $\langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V}^*\rangle = 0$ implies that $\mathcal{N}$ is symmetric in its symplectic indices.

From the properties, eqs. (B.8), one concludes that $\mathcal{V}$ transforms under Kähler transformations as

$$\mathcal{V} \rightarrow e^{-\frac{1}{2}(\lambda-\lambda^*)}\mathcal{V}\,. \tag{B.13}$$

For further details and identities the interested reader can consult the basic references [10, 42–44], the review [11] or ref. [8, 9] whose conventions and results we follow.

C. Quaternionic Kähler geometry

A quaternionic Kähler manifold is a real $4n_H$-dimensional Riemannian manifold HM endowed with a triplet of complex structures $\mathsf{J}^x : T(\mathsf{HM}) \rightarrow T(\mathsf{HM})\,,\ (x = 1,2,3)$ that satisfy the quaternionic algebra

$$\mathsf{J}^x\mathsf{J}^y = -\delta^{xy} + \varepsilon^{xyz}\mathsf{J}^z\,, \tag{C.1}$$

and with respect to which the metric, denoted by H, is Hermitean

$$\mathsf{H}(\ \mathsf{J}^x X,\ \mathsf{J}^x Y\ ) = \mathsf{H}(X,Y)\,, \qquad \forall X,Y \in T(\mathsf{HM})\,. \tag{C.2}$$

This implies the existence of a triplet of 2-forms $\mathsf{K}^x(X,Y) \equiv \mathsf{H}(\ \mathsf{J}^x X, Y)$ globally known as the $\mathfrak{su}(2)$-valued hyperKähler 2-forms.

The structure of a quaternionic Kähler manifold requires an SU(2) bundle to be constructed over HM with connection 1-form A^x with respect to which the hyperKähler 2-form is covariantly closed, i.e.

$$\mathfrak{D}\mathsf{K}^x \equiv d\mathsf{K}^x + \varepsilon^{xyz}\, \mathsf{A}^y \wedge \mathsf{K}^z = 0\,. \tag{C.3}$$

Then if the curvature of this bundle

$$\mathsf{F}^x \equiv d\mathsf{A}^x + \frac{1}{2}\varepsilon^{xyz}\, \mathsf{A}^y \wedge \mathsf{A}^z\,, \tag{C.4}$$

is equal to minus the hyperKähler 2-form

$$\mathsf{F}^x = -\mathsf{K}^x\,, \tag{C.5}$$

the manifold is a quaternionic Kähler manifold as it appears in supergravity.

The SU(2) connection acts on objects with vectorial SU(2) indices, such as the chiral spinors in this article, as follows

$$\mathfrak{D}\xi_I \equiv d\xi_I + \mathsf{A}_I{}^J \xi_J\,, \tag{C.6}$$

$$\mathfrak{D}\chi^I \equiv d\chi^I + \mathsf{A}^I{}_J \chi^J\,. \tag{C.7}$$

The vector SU(2) indices on $\mathsf{A}^I{}_J$ are raised and lowered under complex conjugation as

$$\mathsf{A}^I{}_J = (\mathsf{A}_I{}^J)^*\,. \tag{C.8}$$

Following ref. [10] we put

$$\mathsf{A}_I{}^J \equiv \frac{i}{2}\, \mathsf{A}^x\, (\sigma_x)_I{}^J\,, \tag{C.9}$$

and similarly for the curvature $\mathsf{F}_I{}^J$ where the 3 matrices $(\sigma_x)_I{}^J$ are the Pauli matrices.

The holonomy group of a quaternionic Kähler manifold HM is $\mathrm{Sp}(1)\times\mathrm{Sp}(2n_H)$ where $\mathrm{Sp}(2n_H) \simeq \mathrm{U}(4n_H)\cap\mathrm{Sp}(4n_H,\mathbb{C})$, so that $\mathrm{Sp}(1)\simeq\mathrm{SU}(2)$. It is convenient to use a Vielbein on HM, denoted by

$$\mathsf{U}^{\alpha I} = \mathsf{U}^{\alpha I}{}_u\, dq^u\,, \quad \text{where} \quad u = 1,\ldots,4n_H\,, \tag{C.10}$$

having as 'flat' indices a pair αI consisting of one $\mathrm{Sp}(2n_H)$ index $\alpha = 1,\ldots,2n_H$ and one SU(2) index $I = 1,2$. We shall refer to this object as the Quadbein. This Quadbein is related to the metric H_{uv} by

$$\mathsf{H}_{uv} = \mathsf{U}^{\alpha I}{}_u\, \mathsf{U}^{\beta J}{}_v\, \varepsilon_{IJ}\mathbb{C}_{\alpha\beta}\,, \tag{C.11}$$

where $\varepsilon_{IJ} = -\varepsilon_{JI}$ and $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ are the flat $\mathrm{Sp}(2n_H)$ and SU(2) invariant metrics. It is required that

$$\begin{aligned} 2\, \mathsf{U}^{\alpha I}{}_{(u}\, \mathsf{U}^{\beta J}{}_{v)}\, \mathbb{C}_{\alpha\beta} &= \mathsf{H}_{uv}\varepsilon^{IJ}\,, \\ \mathsf{U}_{\alpha I\, u} \equiv (\mathsf{U}^{\alpha I}{}_u)^* &= \varepsilon_{IJ}\mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta J}{}_u\,. \end{aligned} \tag{C.12}$$

yThe inverse Quadbein $\mathsf{U}^u{}_{\alpha I}$ satisfies

$$\mathsf{U}_{\alpha I}{}^u\, \mathsf{U}^{\alpha I}{}_v = \delta^u{}_v\,. \tag{C.13}$$

For further details and identities see e.g. refs. [10, 45, 46], the review [11] or ref. [25] whose conventions and results we follow and use.

References

[1] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities. II: twisted self-duality of doubled fields and superdualities*, *Nucl. Phys.* **B 535** (1998) 242 [hep-th/9806106].

[2] E. Bergshoeff, J. Hartong and D. Sorokin, *Q7-branes and their coupling to IIB supergravity*, *JHEP* **12** (2007) 079 [arXiv:0708.2287].

[3] P. Meessen and T. Ortín, *An* SL$(2, Z)$ *multiplet of nine-dimensional type-II supergravity theories*, *Nucl. Phys.* **B 541** (1999) 195 [hep-th/9806120].

[4] G. Dall'Agata, K. Lechner and M. Tonin, $D = 10$*,* $N =$ *IIB supergravity: Lorentz-invariant actions and duality*, *JHEP* **07** (1998) 017 [hep-th/9806140].

[5] E.A. Bergshoeff, J. Hartong, T. Ortín and D. Roest, *Seven-branes and supersymmetry*, *JHEP* **02** (2007) 003 [hep-th/0612072].

[6] B.R. Greene, A.D. Shapere, C. Vafa and S.-T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys.* **B 337** (1990) 1.

[7] D.Z. Freedman and P.K. Townsend, *Antisymmetric tensor gauge theories and nonlinear* σ*-models*, *Nucl. Phys.* **B 177** (1981) 282.

[8] P. Meessen and T. Ortín, *The supersymmetric configurations of* $N = 2$*,* $D = 4$ *supergravity coupled to vector supermultiplets*, *Nucl. Phys.* **B 749** (2006) 291 [hep-th/0603099].

[9] J. Bellorín and T. Ortín, *All the supersymmetric configurations of* $N = 4$*,* $D = 4$ *supergravity*, *Nucl. Phys.* **B 726** (2005) 171 [hep-th/0506056].

[10] L. Andrianopoli et al., $N = 2$ *supergravity and* $N = 2$ *super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032].

[11] A. van Proeyen, $N = 2$ *supergravity in* $d = 4, 5, 6$ *and its matter couplings*, lectures given at the Institute Henri Poincaré, Paris, November (2000), online at http://itf.fys.kuleuven.ac.be/~toine/LectParis.pdf.

[12] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged* $N = 2$ *supergravity: Yang-Mills models*, *Nucl. Phys.* **B 245** (1984) 89.

[13] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of* $N = 2$ *supergravity-matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[14] M.K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, *Nucl. Phys.* **B 193** (1981) 221.

[15] E.A. Bergshoeff, M. de Roo, S.F. Kerstan, T. Ortín and F. Riccioni, SL$(2, R)$*-invariant IIB brane actions*, *JHEP* **02** (2007) 007 [hep-th/0611036].

[16] K. Behrndt, D. Lüst and W.A. Sabra, *Stationary solutions of* $N = 2$ *supergravity*, *Nucl. Phys.* **B 510** (1998) 264 [hep-th/9705169].

[17] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in* $N = 2$ *supergravity with* R^2 *interactions*, *JHEP* **12** (2000) 019 [hep-th/0009234].

[18] J. Bellorín, P. Meessen and T. Ortín, *Supersymmetry, attractors and cosmic censorship*, *Nucl. Phys.* **B 762** (2007) 229 [hep-th/0606201].

JHEP05(2008)033

[19] P. Claus et al., $N = 2$ *supergravity lagrangians with vector-tensor multiplets*, *Nucl. Phys.* **B 512** (1998) 148 [hep-th/9710212].

[20] J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, *Phys. Rev. Lett.* **63** (1989) 2443.

[21] B. de Wit and A. Van Proeyen, *Isometries of special manifolds*, hep-th/9505097.

[22] J.J. Atick, A. Dhar and B. Ratra, *Superstring propagation in curved superspace in the presence of background super Yang-Mills fields*, *Phys. Lett.* **B 169** (1986) 54.

[23] P.S. Howe, J.M. Izquierdo, G. Papadopoulos and P.K. Townsend, *New supergravities with central charges and Killing spinors in* $(2+1)$*-dimensions*, *Nucl. Phys.* **B 467** (1996) 183 [hep-th/9505032].

[24] U. Theis and S. Vandoren, $N = 2$ *supersymmetric scalar-tensor couplings*, *JHEP* **04** (2003) 042 [hep-th/0303048].

[25] M. Huebscher, P. Meessen and T. Ortín, *Supersymmetric solutions of* $N = 2$ $D = 4$ *SUGRA: the whole ungauged shebang*, *Nucl. Phys.* **B 759** (2006) 228 [hep-th/0606281].

[26] E. Bergshoeff et al., *Spacetime-filling branes and strings with sixteen supercharges*, *Nucl. Phys.* **B 564** (2000) 29 [hep-th/9812224].

[27] F. Riccioni, *Spacetime-filling branes in ten and nine dimensions*, *Nucl. Phys.* **B 711** (2005) 231 [hep-th/0410185].

[28] E.A. Bergshoeff, M. de Roo, S.F. Kerstan and F. Riccioni, *IIB supergravity revisited*, *JHEP* **08** (2005) 098 [hep-th/0506013].

[29] E.A. Bergshoeff, M. de Roo, S.F. Kerstan, T. Ortín and F. Riccioni, *IIA ten-forms and the gauge algebras of maximal supergravity theories*, *JHEP* **07** (2006) 018 [hep-th/0602280].

[30] F. Riccioni and P. West, *The* E_{11} *origin of all maximal supergravities*, *JHEP* **07** (2007) 063 [arXiv:0705.0752].

[31] E.A. Bergshoeff, I. De Baetselier and T.A. Nutma, E_{11} *and the embedding tensor*, *JHEP* **09** (2007) 047 [arXiv:0705.1304].

[32] E.A. Bergshoeff, J. Gomis, T.A. Nutma and D. Roest, *Kac-Moody spectrum of (half-)maximal supergravities*, *JHEP* **02** (2008) 069 [arXiv:0711.2035].

[33] J. Gomis and D. Roest, *Non-propagating degrees of freedom in supergravity and very extended* G_2, *JHEP* **11** (2007) 038 [arXiv:0706.0667].

[34] E. Bergshoeff, M. de Roo and E. Eyras, *Gauged supergravity from dimensional reduction*, *Phys. Lett.* **B 413** (1997) 70 [hep-th/9707130].

[35] I.V. Lavrinenko, H. Lü and C.N. Pope, *Fibre bundles and generalised dimensional reductions*, *Class. and Quant. Grav.* **15** (1998) 2239 [hep-th/9710243].

[36] N. Kaloper, R.R. Khuri and R.C. Myers, *On generalized axion reductions*, *Phys. Lett.* **B 428** (1998) 297 [hep-th/9803066].

[37] C.M. Hull, *Massive string theories from M-theory and F-theory*, *JHEP* **11** (1998) 027 [hep-th/9811021].

[38] P.M. Cowdall, *Novel domain wall and Minkowski vacua of* $D = 9$ *maximal* SO(2) *gauged supergravity*, hep-th/0009016.

[39] C.M. Hull, *Gauged $D = 9$ supergravities and Scherk-Schwarz reduction*, *Class. and Quant. Grav.* **21** (2004) 509 [hep-th/0203146].

[40] E. Bergshoeff, U. Gran and D. Roest, *Type IIB seven-brane solutions from nine-dimensional domain walls*, *Class. and Quant. Grav.* **19** (2002) 4207 [hep-th/0203202].

[41] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, *Duality of type II 7-branes and 8-branes*, *Nucl. Phys.* **B 470** (1996) 113 [hep-th/9601150].

[42] A. Ceresole, R. D'Auria and S. Ferrara, *The symplectic structure of $N = 2$ supergravity and its central extension*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 67 [hep-th/9509160].

[43] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[44] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kähler geometry?*, *Nucl. Phys.* **B 503** (1997) 565 [hep-th/9703082].

[45] E. Bergshoeff et al., *Superconformal $N = 2$, $D = 5$ matter with and without actions*, *JHEP* **10** (2002) 045 [hep-th/0205230].

[46] E. Bergshoeff et al., *The map between conformal hypercomplex/hyper-Kähler and quaternionic(-Kähler) geometry*, *Commun. Math. Phys.* **262** (2006) 411 [hep-th/0411209].